COLLEGE ALGEBRA

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COLLEGE ALGEBRA

ALTERNATE EDITION

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WASHINGTON UNIVERSITY

THE MACMILLAN COMPANY

NEW YORK 1947

PRINTED IN THE UNITED STATES OF AMERICA

Preface to Regular Edition

This textbook has been prepared for the use of first-year students in colleges and technical schools. The methods of presentation are those that have been found most successful during many years of teaching experience with groups and individuals having varying degrees of preparation. A preliminary multigraphed edition was used in more than twenty sections of students in various divisions of Washington University—the College of Liberal Arts, including pre-business and other pre-professional courses, the Schools of Engineering and Architecture, and the evening classes.

Although the primary objective has been clarity of explanation, the question of logical rigor has been kept constantly in mind, and an effort has been made to distinguish carefully between assumptions and proofs. It is hoped that an appreciation of rigor will be developed in students of the book, particularly those students who later take more advanced courses in mathematics.

The first part of the book constitutes a thorough review of the topics of elementary and intermediate algebra. It was written with the idea in mind that many students have little skill in algebraic manipulation, although it does assume an intelligence level that may reasonably be expected of college or university freshmen. Well prepared classes can cover this part quite rapidly, or even omit it altogether, and proceed to the later portions. Mature students will find enough material for a very complete course containing all of the topics usually taught in College Algebra, also certain additional topics which, because of their importance, it has seemed desirable to include. Those not so well prepared can cover the first part more slowly and then be given such chapters and topics in the latter part of the book as the instructor may deem appropriate.

The arrangement of chapters and topics was decided upon only after members of the Macmillan staff had consulted a considerable number of teachers of the subject in various institutions in different parts of the country. Although the present order is a composite of suggestions, and represents the arrangement which seemed most desirable to the majority, the individual chapters have been rendered to a large extent independent by means of cross references and a certain amount of repetition of important connecting ideas. Because of this, the instructor who prefers a different sequence of chapters, or who desires to omit certain material, will find that the book can readily be adapted to his needs.

The book contains a liberal supply of selected and graded exercises. Answers to the odd-numbered exercises are printed at the back of the book, answers to the even-numbered exercises are printed in a separate pamphlet. This pamphlet is available to students upon request of the instructor.

Among the special features may be mentioned the treatment of determinants. This follows the method outlined by G. D. Birkhoff in his article "Determinants," in the *Encyclopedia Britannica*. It is believed to be somewhat simpler than the usual method based upon the idea of inversions in order. My thanks are due to my colleague, Professor H. R. Grummann, for calling my attention to this method.

The chapter on the theory of equations contains more material than is to be found in many texts on college algebra. One important feature is the inclusion of Descartes' rule of signs in extended form, which states that the number of positive roots is equal to the number of variations in sign *or is less by an even number*. In this form it is much more useful and but slightly more difficult to prove. In addition to Horner's method for the solution of equations, another approximate method, which is essentially linear interpolation, is given. This alternate method, in addition to its simplicity, has the advantage of being applicable to any type of equation. The material is so arranged that either method may be omitted at the option of the instructor.

Because of its usefulness, the binomial formula for other than positive integral exponents is given, without proof, however.

A decided simplification is made in the rules for determining the characteristic of a logarithm and for pointing off an antilogarithm, whereby the four rules ordinarily given are replaced by a single rule. The question of the accuracy of results obtained by using approximate numbers is discussed.

In connection with the chapter on compound interest and annuities there are given tables of compound amounts, present values, amounts and present values of annuities. Since anyone having much to do with financial transactions will use tables of this sort, it seems desirable for the student to learn to employ such tables in the solution of problems.

The application of probability to mortality tables is emphasized in the chapter on probability.

Because of the recent rapid growth of the subject of statistics and because many students may later be engaged in statistical work, it has seemed advisable to include an elementary chapter on finite differences. This subject, long omitted from books on algebra, deserves to be included again.

I wish to take this opportunity to thank my colleagues in Washington University for using the preliminary multigraphed edition of the book and for making various suggestions for its improvement. I wish also to express my appreciation to The Macmillan Company for the very valuable editorial assistance which they have rendered during the process of revision and publication. The revision of the multigraphed edition was critically read by three of their advisers, and after another very thorough revision, in which the criticisms of these advisers were taken into account and their suggestions incorporated, it was again read by three advisers, after which further revisions were made.

P. R. R.

St. Louis, Mo.
February, 1940.

Preface to the Alternate Edition

This edition has been prepared to provide fresh problem material. The exercises are all new with the exception of several which are of a theoretical nature. They have been increased in number by about ten per cent. They have been carefully chosen so as to bring out the various topics treated and to illustrate a wide variety of applications of algebra. It is believed that they will furnish ample material for drill in algebraic manipulation and also that many of them are sufficiently thought-provoking to prove challenging and stimulating to the student. The text matter has been retained intact save for a few very minor changes.

P. R. R.

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CHAPTER I

Review of Elementary Algebra

1. Introduction.

We shall recall in this chapter certain principles which are the results of the fundamental assumptions of algebra. Exercises are provided for illustrating the application of these principles and for the purpose of reviewing various matters of algebraic manipulation.

2. Fundamental assumptions.

Some of the fundamental assumptions concerning real numbers, both positive and negative, are as follows:

The sum of two numbers is the same in whatever order the numbers are added. That is,

$$a + b = b + a.$$

This is called the **commutative law** for addition.

The sum of three or more numbers is the same in whatever way the numbers are grouped. Thus,

$$a + b + c = (a + b) + c = a + (b + c).$$

This is called the **associative law** for addition.

Its usefulness may be illustrated by the following example from arithmetic: $36 + 25 + 64$. Here it is easiest to combine the first and third numbers, obtaining 100, and then, making use of the commutative law, to add the 25, getting the final result, 125. Its usefulness in algebra is even greater.

The product of two numbers is the same in whatever order the numbers are multiplied. That is,

$$ab = ba.$$

This is called the **commutative law** for multiplication.

The product of three or more numbers is the same in whatever way the numbers are grouped. Thus,

$$abc = (ab)c = a(bc).$$

This is the **associative law** for multiplication.

For example, in evaluating the product $125 \cdot 173 \cdot 8$, we find it quite simple to multiply the first and third factors together, obtaining 1000, and as a final step to get $1000 \cdot 173 = 173{,}000$, whereas any other manner of grouping the factors would require more work.

The product of a number and the sum of other numbers is the same as the sum of the products obtained by multiplying each of the other numbers by the first number. In symbols,

$$a(b + c) = ab + ac.$$

This is the **distributive law** for multiplication with respect to addition.

As a numerical illustration we note that $6(3 + 2) = 6 \cdot 3 + 6 \cdot 2$.

Other fundamental assumptions are the following **axioms:**

Quantities equal to the same quantity or to equal quantities are equal to each other.

If equals are added to equals the sums are equal.

If equals are subtracted from equals the remainders are equal.

If equals are multiplied by equals the products are equal.

*If equals are divided by equals the quotients are equal.**

* Division by zero is excluded. See section 4.

A quantity may be substituted for its equal in any expression.

Most of the axioms concerning equal quantities may be combined into a single axiom: *If the same operation is performed on equals, the results are equal.*

3. Subtraction.

To subtract b from a is to find a number c such that $a = b + c$.

4. Division.

To divide a by b is to find a number c such that $a = bc$.

By this definition, if a is any number, and b is zero, then c is the number for which $a = 0 \cdot c$. But $0 \cdot c = 0$, and thus there is no number c for which $a = 0 \cdot c$. An exception would occur if a itself were zero. Then c could be any number whatever.

Since division by zero is thus either impossible or ambiguous, it is always excluded from all algebraic operations. With this sole exception, each of the four operations (addition, subtraction, multiplication, and division) is always possible and always yields a unique result.

5. Signs.

The sign preceding the product or quotient of two quantities is plus (+) or minus (−) according as the quantities have like or unlike signs.

Examples.

$$a \cdot b = ab, \quad (-3) \cdot (-2) = 6, \quad \frac{6x}{2x} = 3, \quad \frac{-8}{-2} = 4, \quad \frac{8}{-2} = -4.$$

How can the foregoing rule be extended to the multiplication and division of three or more quantities?

6. Symbols of grouping.

Several numbers are frequently grouped to indicate that they are to be considered together as a single number. The symbols of grouping are: **parentheses** ( ), **brackets** [ ], **braces** { }, and **vinculum** ‾

To remove symbols of grouping preceded by a minus sign, multiply each term * within the symbols of grouping by -1.

Examples.

$$4x - (3y - z + 2) = 4x - 3y + z - 2.$$
$$7a - [6b - (c - 2d) + e] = 7a - 6b + (c - 2d) - e.$$

If an expression is to be enclosed within symbols of grouping preceded by a minus sign, each term of the expression must be multiplied by -1.

Symbols of grouping preceded by a plus sign may be removed or inserted whenever desired, without affecting the value of an expression.

Examples.

$$2x + (y - 2) = 2x + y - 2.$$
$$a - b + c = a + (-b + c).$$

A quantity multiplying or dividing an expression within symbols of grouping multiplies or divides every term within the symbols of grouping.

Examples.

$$2 - 3(5x - 2y) = 2 - 15x + 6y.$$
$$x + 3(2y - z) = x + 6y - 3z.$$

The foregoing rules apply when symbols of grouping are contained within other such symbols.

* When an expression is made up of parts connected by plus or minus signs, each part, taken with the sign immediately preceding it (+ being understood for the first part if no sign precedes it), is called a **term**.

Example.

$$3a - \{b + 2[x - 4y - (c - d)] - 5\}$$
$$= 3a - \{b + 2[x - 4y - c + d] - 5\}$$
$$= 3a - \{b + 2x - 8y - 2c + 2d - 5\}$$
$$= 3a - b - 2x + 8y + 2c - 2d + 5.$$

EXERCISES I. A

Remove all symbols of grouping and combine like terms:

1. $10 - [5 - (7 - 3) + (8 - 2)]$.
2. $10 - 3[5 - 2(7 - 3) + 3(8 - 2)]$.
3. $2a - [a - (2b - c) + a(b - c - 3)]$.
4. $5x - \{3a - 2x[a - b - \overline{(3 - 2y)}] - 3y\}$.
5. $4\{5p - 2[q - 3(2p + 3q - \overline{r - s}) + 7p]\}$.
6. $a + 6b\{5 - a - 2[3a - 4(c + 2d) - 2]\}$.
7. $4(2a - 3b) - 2\{3a - 2b + 5[2a + 3b - (2a - 3)]\}$.
8. What is the value of the following expression if $x = 3$, $y = -2$?

   $4(3x - 2y + 1) - 3\{x - 2y[3x - 4 + 2(1 - x)] - 2xy\}$.

Place the last three terms of the following expressions in parentheses preceded by a minus sign:

9. $z - x - 2y - 3$.
10. $p - q - r + s$.
11. $a + 2b - 3c - 4d$.
12. $-3m - 2n + 4h + 7k$.

7. Exponents.

The symbol a^3 means $a \cdot a \cdot a$. In general, if n is a positive whole number, then a^n means $a \cdot a \cdots a$ (n factors), and is called the **nth power** of a, n being the **exponent** of the power.

The symbol $\sqrt[n]{a}$ (**nth root** of a) is a number which raised to the nth power will give a. The integer n is called the **index** of the root. (See section 33.)

It follows that

$$a^3 \cdot a^2 = (a \cdot a \cdot a) \cdot (a \cdot a) = a^{3+2} = a^5,$$
$$\frac{a^5}{a^3} = \frac{a \cdot a \cdot a \cdot a \cdot a}{a \cdot a \cdot a} = a^{5-3} = a^2,$$

$$(a^3)^4 = a^3 \cdot a^3 \cdot a^3 \cdot a^3 = a^{4\cdot 3} = a^{12},$$
$$\sqrt[3]{a^{12}} = a^{12/3} = a^4.$$

It also follows, from the commutative law for multiplication, that

$$a^2 \cdot b^2 = (a \cdot b)^2.$$

In general, if m and n are positive whole numbers,

$$a^m a^n = a^{m+n},$$
$$\frac{a^m}{a^n} = a^{m-n}, \quad m > n, \quad a \neq 0,$$
$$\frac{a^m}{a^n} = \frac{1}{a^{n-m}}, \quad m < n, \quad a \neq 0,$$
$$(a^m)^n = a^{mn}.$$

EXERCISES I-B

Perform the indicated operations:

1. $a^2 \cdot a^6$.
2. $2^3 \cdot 2^4$.
3. $x^2y^3 \cdot x^5y^4$.
4. $3a^2b \cdot 2ab^2$.
5. $4p^4 \cdot 3p^3$.
6. $(-2x^5)(-5x^2)$.
7. $(3a^3)^2$.
8. $-x^3(-xy)(-y)^2$.
9. $-(4x^2)^2$.
10. $(-4x^2)^2$.
11. $-(4x^2)^3$.
12. $(-4x^2)^3$.
13. $\dfrac{9x^9}{3x^3}$.
14. $\dfrac{4a^4}{2a^2}$.
15. $\dfrac{14x^3y^2z^7}{4xy^2z^3}$.
16. $\dfrac{-18ab^2c^4d^6}{24ab^2c^3d^4}$.
17. $\dfrac{m^4}{m^{10}}$.
18. $\dfrac{6a^6b^2c^3d^5}{15ab^4c^4d^3}$.
19. $\dfrac{15r^{15}s^5t^3}{-10r^{10}s^5t^5}$.
20. $\dfrac{4(u^2v^3w)^3}{8(uv^2w^3)^2}$.
21. $(a^8 \div a^3) \cdot a^4$.
22. $a^8 \div (a^3 \cdot a^4)$.

23. $(a^8 \cdot a^3) \div a^4$.
24. $a^8 \cdot (a^3 \div a^4)$.
25. $(12x^6 \div 2x^2) \cdot 4x^4$.
26. $12x^6 \div (2x^2 \cdot 4x^4)$.
27. $(12x^6 \cdot 2x^2) \div 4x^4$.
28. $12x^6 \cdot (2x^2 \div 4x^4)$.

8. Multiplication of polynomials.

An algebraic term (see page 6, footnote) is said to be **integral and rational** in certain literal numbers if it is the product of positive integral powers of these numbers multiplied by a factor not involving them or if it does not involve them. Thus, the terms $5x^2y^3$, $-3x$, $x^2\sqrt{3}$, and 7 are integral and rational in x and y. The terms $2/x$ and $x^2\sqrt{y}$ are not integral and rational in x and y. (The last is integral and rational in x but not in y.)

An algebraic expression is called a **monomial** if it is composed of just one term, a **binomial** if it is composed of just two terms, and a **trinomial** if it is composed of just three terms. An expression composed of more than one term is called a **polynomial.**

An **integral rational polynomial** is one in which each term is integral and rational. Unless otherwise stated, all polynomials to which we refer will be understood to be integral and rational in all literal numbers involved.

NOTE. It is sometimes convenient to classify an integral rational monomial as a polynomial.

Example.

Multiply $2a^2 - 3ab + b^2$ by $a^2 + 2ab - 5b^2$.

SOLUTION.

$$\begin{array}{lllll}
2a^2 & - 3ab & + b^2 & & \\
a^2 & + 2ab & - 5b^2 & & \\
\hline
2a^4 & - 3a^3b & + a^2b^2 & & \\
 & 4a^3b & - 6a^2b^2 & + 2ab^3 & \\
 & & - 10a^2b^2 & + 15ab^3 & - 5b^4 \\
\hline
2a^4 & + a^3b & - 15a^2b^2 & + 17ab^3 & - 5b^4
\end{array}$$

CHECK. Let $a = 3, b = 2$.

$$2a^2 - 3ab + b^2 = 18 - 18 + 4 = 4$$
$$a^2 + 2ab - 5b^2 = 9 + 12 - 20 = 1$$
$$\text{Product} = 4.$$
$$2a^4 + a^3b - 15a^2b^2 + 17ab^3 - 5b^4$$
$$= 162 + 54 - 540 + 408 - 80 = 4.$$

The first line below the first horizontal rule, viz., $2a^4 - 3a^3b + a^2b^2$, is the product of a^2 and $2a^2 - 3ab + b^2$, and so on. Similar terms, such as $-3a^3b$ and $4a^3b$, have been placed in columns and added. This is permissible, since the order in which numbers are added is immaterial.

The work may be checked algebraically by multiplying with multiplier and multiplicand interchanged, or by dividing (see next section) the product by either of the polynomials to obtain the other. Checking by the substitution of numbers does not prove that the algebraic work is correct. If the numbers check it is probable that the work is correct; on the other hand, if the numbers fail to check there is certainly a mistake.

EXERCISES I. C

Multiply:

1. $2x^2 - 3x + 7$ by $x + 3$.
2. $3x^2 + 5x - 4$ by $x - 5$.
3. $6a^2 - 7a - 3$ by $2a + 5$.
4. $5k^2 - 6k + 1$ by $3k - 4$.
5. $2m^2 - 8m - 3$ by $2m^2 - 3$.
6. $4x^2 - 3x + 7$ by $3x^2 - 2x - 4$.
7. $2x^2 - 3xy + 5y^2$ by $x^2 + 2xy - 3y^2$.
8. $x^3 - 2x^2y - y^3$ by $x^2y - xy^2 + 2y^3$.
9. $3x^2 - 2x^3 - 5x + 2$ by $x - x^2 + 3$.

   SUGGESTION. Rearrange both expressions according to descending powers of x.

10. $2x^2 - x + 5$ by $3 - 2x$.
11. $3a^2 - 7a^3 + 10a - 2$ by $a^2 - 2a^3 - 5a + 4$.
12. $a^2b + 2a^3 - ab^2$ by $2ab + b^2 - a^2$.
13. $3x - 2y - 4z + w$ by $2x + 3y - z - 5w$.

Square:

14. $x^2 - 2y^2$.

15. $2a^3 + 3b^2$.

16. $x + y + z$.

17. $a + b - c$.

Perform the indicated multiplications:

18. $(a - 3)(2a + 5)(a^2 - 5a + 6)$.

19. $(2x^2 - 3x + 2)(x^2 + 2x - 1)(x^2 + 3)$.

Perform the indicated operations:

20. $(x - y)^2(x + y)$.

21. $(x - y)^3$.

22. $(2a + 3b)^2(2a - 3b)^2$.

23. $(2a - 3b)^2(3a - 2b)^2$.

9. Division of polynomials.

Polynomials may be divided as shown in the following example:

Example.

Divide $5x^2 - 2x^3 + 3$ by $x^2 - 4 - 2x$.

SOLUTION. Arrange both polynomials according to descending powers of x. The work is shown below.

$$\begin{array}{rl} & -2x + 1 \;(= \text{quotient}) \\ (\text{divisor} =)\; x^2 - 2x - 4) & -2x^3 + 5x^2 \qquad\quad + 3 \;(= \text{dividend}) \\ & \underline{-2x^3 + 4x^2 + 8x} \\ & \qquad x^2 - 8x + 3 \\ & \qquad \underline{x^2 - 2x - 4} \\ & \qquad\quad - 6x + 7 \;(= \text{remainder}) \end{array}$$

We find, by dividing the first term of the divisor into the first term of the dividend, that the first term in the quotient is $-2x$ (i.e., $-2x^3/x^2 = -2x$). Multiply the divisor $x^2 - 2x - 4$ by $-2x$, obtaining the polynomial $-2x^3 + 4x^2 + 8x$, which is written below the dividend and subtracted from it. Continue the process until the remainder is of lower degree * than the divisor.

* The **degree** of a polynomial consisting of positive integral (i.e., whole-number) powers of x is the exponent of the highest power of x occurring in the polynomial. Thus, the polynomial $2x^3 - 5x^2 + 4$ is of degree three.

Division may be checked by using the relation

$$\textbf{Dividend} = \textbf{Quotient} \times \textbf{Divisor} + \textbf{Remainder} \qquad (1)$$

either algebraically or by substituting numbers. A number which makes the divisor equal to zero must not be used. (See section 4.)

We shall check the foregoing division by letting $x = 2$. We find

$$\begin{aligned} \text{Dividend} &= -2x^3 + 5x^2 + 3 = -16 + 20 + 3 = 7. \\ \text{Divisor} &= x^2 - 2x - 4 = 4 - 4 - 4 = -4. \\ \text{Quotient} &= -2x + 1 = -4 + 1 = -3. \\ \text{Remainder} &= -6x + 7 = -12 + 7 = -5. \end{aligned}$$

Substituting in (1), we get

$$7 = -3(-4) - 5, \qquad \text{or} \qquad 7 = 7.$$

EXERCISES I. D

Divide:

1. $x^3 - x^2 - 8x - 4$ by $x + 2$.
2. $6x^3 - 5x^2 - 20x + 21$ by $2x - 3$.
3. $10x^3 + 13x^2 - 17x - 21$ by $5x^2 - x - 7$.
4. $8x^3 + 14x^2 - 31x + 12$ by $2x^2 + 5x - 4$.
5. $7x^2 + 4 - 6x^3 - 10x$ by $2x - 1$.
6. $14 + 12x^3 - 7x^2 - 55x$ by $4x^2 + 7x - 2$.
7. $12a^3 + 2a^2 + 41a + 30$ by $3a + 2$.
8. $12r^3 - 46r^2 + 240r - 175$ by $2r^2 - 6r + 35$.
9. $15x^3 - 8x^2y - 6xy^2 + 4y^3$ by $5x^2 - 6xy + 2y^2$.
10. $3x^2 - 2x + 4$ by $x + 3$.
11. $x^3 - 7x^2 + 5x - 2$ by $x - 2$.
12. $6x^3 - 5x^2 - 8x + 4$ by $2x - 3$.
13. $12x^3 - 6x^2 + x + 2$ by $3x^2 + 4$.
14. $7x^3 - 25x^2 - 2x + 8$ by $x^2 - 3x - 6$.
15. $x^4 - 2x^2 - 100x$ by $x - 5$.
16. $x^2 - y^2$ by $x - y$.
17. $x^2 + y^2$ by $x - y$.

18. $x^3 - y^3$ by $x - y$.

19. $x^3 + y^3$ by $x - y$.

20. $x^3 + y^3$ by $x + y$.

21. $x^5 + y^5$ by $x + y$.

22. $6x^4 - 2x^3 - 15x^2 + 14x - 3$ by $3x^2 - 4x + 1$.

Perform the indicated operations:

23. $[(a^3 - b^3) \div (a - b)](a + b)$.

24. $(a^4 - b^4) \div [(a + b)(a - b)]$.

10. Constants and variables.

A symbol which, throughout a discussion, does not change in value is called a **constant.** (There are, of course, some constants, such as 3, -2, and π, which never change in value.)

A symbol which may change in value during a discussion is called a **variable.**

In the formula for the area of a triangle,

$$A = \frac{1}{2}bh,$$

$\frac{1}{2}$ is a never changing constant. If in a certain problem we regard only the base b of the triangle as fixed in value, then for that problem, b is a constant, and the altitude h and the area A are variables.

11. Functions.

When two variables are so related that to each of a set of values of one there correspond one or more values of the other, the second variable is said to be a **function** of the first. The first variable is called the **independent** variable, the second the **dependent** variable. For example, if $y = x^2$, then y is a function of x, and we may regard x as the independent variable, to which values are arbitrarily assigned, and y as dependent on x.

This type of relation is often written in the form

$$f(x) = x^2,$$

which may be read "the f function of x equals x^2," or, more briefly, "f of x equals x^2." If $f(x) = x^2$, then $f(a) = a^2$, $f(3) = 3^2 = 9$, $f(-3) = (-3)^2 = 9$, and so on.

In another problem or discussion we might have $f(x) = 2x + 3$ or $f(x) = 1/x$. That is, $f(x)$ does not always represent the same function of x. Nor is f the only letter used to represent functions; we may have $F(x)$, $g(x)$, $\varphi(x)$, etc.

The area of a circle is obviously a function of its radius, and we may write $A = f(r)$. In this instance we know the form of the function, namely, $f(r) = \pi r^2$. Similarly, the edge of a cube is a function of its volume. Here $f(V) = \sqrt[3]{V}$.

Often one variable is a function of two or more other variables. For example, the volume of a right circular cone is a function of its base and its altitude, since $V = \frac{1}{3}\pi r^2 h$. This may be written

$$V = f(r,h) = \frac{1}{3}\pi r^2 h.$$

If the radius is 2 and the altitude 3, we have

$$V = f(2,3) = \frac{1}{3}\pi \cdot 2^2 \cdot 3 = 4\pi.$$

EXERCISES I. E

1. Given $f(x) = 3x + 2$; find $f(0)$, $f(1)$, $f(2)$, $f(-1)$, $f(-2)$, $f(a^2)$, $f(-a)$, $f(1 - y)$.
2. Given $f(x) = 2 - x$; find $f(0)$, $f(1)$, $f(2)$, $f(-5)$, $f(y)$, $f(2y)$, $f(2a + 3)$.
3. Given $f(x) = x^2 - 3x + 4$; find $f(0)$, $f(1)$, $f(5)$, $f(-5)$, $f(100)$.
4. Given $f(x) = \dfrac{x - 1}{x + 2}$; find $f(0)$, $f(1)$, $f(2)$, $f(-1)$, $f(5)$, $f(-5)$.

5. Given $f(x) = x^2 - 4$, $g(x) = 2x - 3$; find $f(3) + g(2)$, $f(1)g(4), \frac{f(-6)}{g(1)}, \frac{f(0)}{f(2) + g(3)}, \frac{[f(3)]^2}{6 - g(6)}, f(2a)g(a^2)$.

6. Given $f(x) = x - 1$; show that $\frac{f(a^2)}{f(a) + 2} = f(a)$.

7. Given $f(x) = 2x - 3$; find $f(a + 2)[f(a) + 2]$.

8. Given $f(x,y) = x^2y - \frac{x}{y}$; find $f(4,2), f(-3,1), f(0,6), f(6,-3)$, $f(a,a), f(2k,k)$.

9. Given $F(x,y) = \frac{x^2 - 2xy + 3y^2}{2x + y}$; find $F(2,1)$, $F(-3,-2)$, $F(a,0), F(0,b), F(a,a)$.

Express the following in functional notation and then give the particular form of the function:

10. The radius r of a circle as a function of its circumference c.

Solution. $r = f(c) = \frac{c}{2\pi}$.

11. The area A of a circle in terms of its diameter d.
12. The area A of a square as a function of its diagonal d.
13. The area A of a circle as a function of its circumference c.
14. The cost C, in dollars, of G gallons of gasoline at 20 cents a gallon.
15. The distance d, in miles, traveled by an automobile going at the rate of 60 miles an hour, as a function of the time t in hours.
16. The area A of a rectangle as a function of its length l and width w.
17. The lateral surface L of a right circular cylinder as a function of its radius r and its height h.
18. The base b of a triangle as a function of its area A and its altitude h.
19. The altitude h of a right circular cone in terms of its volume V and its radius r.
20. A box is constructed by cutting square corners from a piece of cardboard 12 by 12 inches and bending up the sides. Express the volume V of the box as a function of the side x of the square cut out.

12. Graphs.

Let us take two straight lines $X'X$ and $Y'Y$ intersecting at right angles at the point O. (See Fig. 1.) On each we choose a unit of measurement and mark off a scale with the zero point at O. Positive numbers are to the right on the horizontal line $X'X$, upward on the vertical line $Y'Y$; negative numbers are to the left on $X'X$, downward on $Y'Y$. $X'X$ is called the **x-axis**, $Y'Y$ the **y-axis**; the point O is called the **origin**.

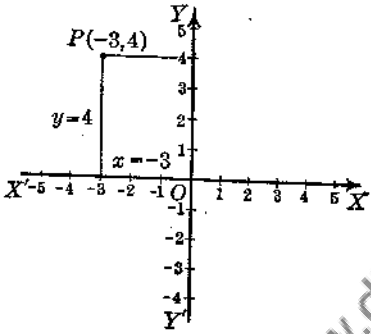


Fig. 1

Now take any point P. The distance of the point from the y-axis is called the **abscissa** of the point and is denoted by x, its distance from the x-axis is called its **ordinate** and is denoted by y. The abscissa and ordinate together are called the **coordinates** (more specifically, the **rectangular coordinates**) of the point. The point P, whose abscissa is x and whose ordinate is y, is denoted by (x,y) or $P(x,y)$, the abscissa always being named first.

According to the above convention, if the point is to the right of the y-axis its abscissa is positive, if it is to the left its abscissa is negative. If it is above the x-axis its ordinate is positive, if below its ordinate is negative. The point $P(-3,4)$ in Fig. 1 has the abscissa $x = -3$ and the ordinate $y = 4$.

The process of locating and marking a point whose coordinates are given is called **plotting** the point. Plotting is facilitated by the use of coordinate, or graph, paper. (See Fig. 2.)

Let us now consider the function

$$y = 2x + 3. \qquad (1)$$

By assigning values to x and finding the corresponding

$y = 2x + 3$

x	y
-3	-3
-2	-1
-1	1
0	3
1	5
2	7
3	9

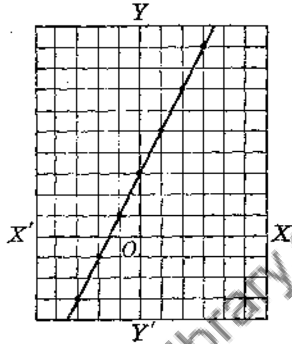


FIG. 2

values of y we can construct a table such as the one accompanying the figure. If the pairs of numbers in this table are taken as coordinates of points and these points are plotted they will be found to lie on a straight line. (See Fig. 2.) The line is called the **graph** of the function $f(x) = 2x + 3$.

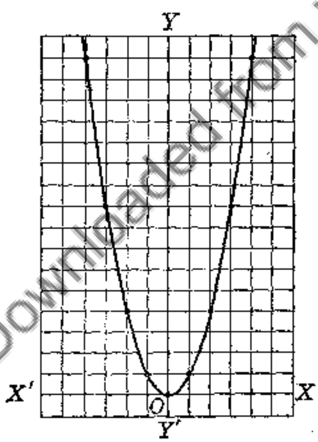


FIG. 3

$y = x^2$

x	y
-4	16
-3	9
-2	4
-1	1
0	0
1	1
2	4
3	9
4	16

Proceeding in the same way with the function

$$y = x^2, \qquad (2)$$

we find that the points which we obtain will lie on a curve (Fig. 3).

If the relation between x and y is given in the form

$3x + 2y = 6$, for example, in which neither variable is expressed explicitly as a function of the other, we may solve for one of them before attempting to construct the graph. Thus, in the above equation we should find

$$2y = 6 - 3x, \qquad y = 3 - \frac{3x}{2},$$

before obtaining values and plotting.

If we had $y^2 - x + 4 = 0$, it would be better to solve for x, getting $x = y^2 + 4$, and then to give values to y, for if we solve for y, obtaining $y = \pm\sqrt{x - 4}$, we should have to extract square roots.

EXERCISES I. F

Plot the points:

1. $(4,7)$, $(4,-7)$, $(-4,7)$, $(-4,-7)$, $(7,4)$, $(7,-4)$, $(-7,4)$, $(-7,-4)$.
2. $(3,0)$, $(0,3)$, $(-3,0)$, $(0,-3)$, $(0,0)$, $(3,3)$, $(3,-3)$, $(-3,3)$.
3. $(1.5,-2.3)$, $(\pi,-2)$, $(3,-2\pi)$, $(\frac{1}{3},\frac{1}{2})$, $(\frac{1}{3},-\frac{1}{2})$.
4. Find the distance from the origin to each of the following points:

$$(4,-3),\ (5,12),\ (-6,8),\ (15,8),\ (-7,-24).$$

Suggestion. Use the theorem of Pythagoras concerning the hypotenuse and sides of a right triangle.

Draw the graphs of the following equations:

5. $y = x - 3$.
6. $y = 2x + 3$.
7. $x = 2y + 3$.
8. $y = 3 - 2x$.
9. $2x + 3y = 12$.
10. $2x - 3y = 12$.
11. $5x - 6y + 15 = 0$.
12. $5x + 6y = 0$.
13. $y = \frac{1}{2}x^2 - 2$.
14. $x^2 + 2y - 8 = 0$.
15. $y = 0.2x^2 - 4x + 3$.
16. $y = 3x - x^2$.
17. $y = \frac{1}{2}x^3 - 3x + 1$
18. $y = \frac{1}{2}x^3 - x^2 + 3$.
19. $y = 9 - x^2$.
20. $x^2 + y^2 = 100$.

MISCELLANEOUS REVIEW EXERCISES I. G

Remove all signs of grouping and combine like terms:

1. $2a - (3a - 2b) + (5a - 6b)$.
2. $6x - [2x - 4y - (x + 4y - 3) - 7]$.
3. $20a - 2\{3a - 4[5a - 2b - (4 - a)] + 7b\}$.
4. $a(b + 2c) + 3b(4a - 5c) - 6c(7a - 8b)$.
5. $x - 3\{2y - [7x - 4(x - y - z) - 3x(y - z - 2)]\}$.
6. Find the value of the expression in the preceding exercise when (a) $x = 2, y = 4, z = 3$; (b) $x = 5, y = -4, z = -7$.

Perform the indicated operations:

7. $2a^2 \cdot 3ab^3 \cdot 4a^2b^4c^3$.
8. $(2a)^2(3ab)^2(4a^2b^4c^3)^2$.
9. $132x^8y^6z^3 \div 11x^3y^2$.
10. $91x^5yz^{10} \div 7x^2y^5z^5$.
11. $(x^2 + 3x - 7) \cdot (2x^2 - 5x + 2)$.
12. $(4x^2 - 5xy + 6y^2) \cdot (3x^2 - xy + 2y^2)$.
13. $(3x - 2) \cdot (2x - 5) \cdot (x^2 - x - 2)$.
14. $(2x + 5) \cdot (x^2 - 5x + 6) \cdot (2x^2 + 3x - 4)$.
15. $(1 - x^2 + 2x) \cdot (3x - 5 + x^2)$.
16. $(2x^3 - 17x^2 + 38x - 15) \div (x - 5)$.
17. $(12x^3 + 11x^2 + 11x + 20) \div (4x + 5)$.
18. $(6x^4 + 5x^3 - 29x^2 - 2x + 20) \div (3x^2 - 2x - 4)$.
19. $(10x^4 - 39x^3 - 13x - 12) \div (5x^2 - 2x + 3)$.
20. $(12a^4 + 6a^3 - 16a^2 - 9a - 3) \div (2a^2 - 3)$.
21. Given $f(x) = 2x^2 - 3x + 4$; find $f(0)$, $f(1)$, $f(-1)$, $f(-2)$, $f(3) \cdot f(-4)$, $f(4) \div f(2)$, $f(2a)$, $f(3a)$, $f(a + 2)$.
22. Given $F(x) = (x - 5)(x + 3)$; find $F(0)$, $F(5)$, $F(-3)$, $F(2) \cdot F(3)$, $F(2) \div F(3)$, $F(2m)$.

Plot:

23. $y = 2x - 5$.
24. $y = 5 - 2x$.
25. $y = \frac{1}{2}x + 1$.
26. $x = 2y - 5$.
27. $3x + 4y = 12$.
28. $3x - 4y = 12$.
29. $3x - 4y = 0$.
30. $y = x^2 - 1$.
31. $y = 1 - x^2$.
32. $y = x^2 - 5x + 6$.
33. $y = x^2 + 5x - 6$.
34. $y = 6 - 5x - x^2$.
35. $y = 4x - x^2$.
36. $y = \frac{1}{2}x^3$.

CHAPTER II

Linear Equations

13. Equations.

An **equation** is a statement that two expressions are equal, the two expressions being called the **members,** or **sides,** of the equation. An equation whose members are equal for all permissible values of the variables involved is called an **identical equation,** or simply an **identity.** An equation whose members are equal for certain values of the variables involved, but not for all permissible values, is called a **conditional equation.** The word "equation," used alone, will ordinarily be interpreted to mean "conditional equation." Any number or set of numbers which, when substituted for the variables, makes the members of an equation equal to each other, is called a **solution** of the equation and is said to **satisfy** the equation. Identities are satisfied by all values of the variables which they contain.

The equations

$$2(x - 3) = 2x - 6, \qquad (x + y)^2 = x^2 + 2xy + y^2,$$

$$\frac{x^2 + 2}{x - 1} = x + 1 + \frac{3}{x - 1}$$

are identities, since they are satisfied by all permissible values of the variables. (Note that in the last identity, 1 is not a permissible value for x; the members are not defined for $x = 1$, since this involves division by zero.)

The equation $x + 1 = 5$, on the other hand, is a conditional equation, since it is satisfied only by $x = 4$.

An identity may be written with the sign $\equiv$ instead of $=$. Thus,

$$2(x - 3) \equiv 2x - 6.$$

The symbol $\equiv$ is read "is identically equal to."

14. Linear equations in one unknown.

The **degree** of a term containing any number of variables is the sum of the exponents of the variables appearing in that term, it being understood that these exponents are positive whole numbers. (See section 18.) In the expression $5x + 2 - 6x^3y$ the first term is of degree 1, the second of degree 0 (since $x^0 = 1$,* a constant is said to be of degree 0, that is, $2 = 2x^0$), the third of degree 4 (it is of degree 3 in x alone and of degree 1 in y alone, but of degree 4 when both variables are considered). The expression $x^3 + y^2$ is of degree 3 in x and y.

A **linear equation** is one of first degree in *all* of the unknowns involved. Terms of degree zero—that is, constant terms—may also be present. Thus, $2x - 3 = 0$, $2x - 5y + 7 = 0$, $x + y - 2z = 8$ are linear equations. The equation $2x - 3xy + 6 = 0$ is not linear, because the term $-3xy$ is of second degree.

In solving a linear equation containing a single unknown quantity we perform the same operations on both sides of the equation, with the ultimate purpose of getting the unknown alone on one side of the equation.

Example 1.

Solve: $$\frac{2}{3}x - 3 = 4x + 7.$$

SOLUTION. Multiply both sides by 3 to clear of fractions,

* See section 34.

obtaining

$$2x - 9 = 12x + 21.$$

Add 9 to, and subtract $12x$ from, both sides. This gives

$$-10x = 30.$$

Finally, divide both sides by -10, obtaining

$$x = -3.$$

The solution -3 is called the **root** of the equation, which means that it satisfies the equation. Roots should always be checked. Thus,

$$\frac{2}{3}(-3) - 3 = 4(-3) + 7,$$

$$-5 = -5.$$

Example 2.

Solve for x: $$2a + 3x = 5a + 6.$$

SOLUTION.

$$3x = 3a + 6,$$
$$x = a + 2.$$

CHECK.

$$2a + 3(a + 2) = 5a + 6,$$
$$2a + 3a + 6 = 5a + 6,$$
$$5a + 6 = 5a + 6.$$

Many problems have to be translated into mathematical equations before they can be solved conveniently.

Example 3.

How many ounces of pure silver must be added to 120 ounces of alloy 60 per cent pure to produce an alloy 70 per cent pure?

SOLUTION. Let x = no. of oz. of pure silver that must be added. Then,

$$\begin{aligned}
120 + x &= \text{total no. of oz. in new alloy.}\\
60\% \text{ of } 120 &= 0.60 \times 120 = 72\\
&= \text{no. of oz. of pure silver in old alloy.}\\
72 + x &= \text{no. of oz. of pure silver in new alloy.}\\
\frac{72 + x}{120 + x} &= 70\% = 0.70,\\
72 + x &= 0.70(120 + x) = 84 + 0.7x,\\
x - 0.7x &= 84 - 72,\\
0.3x &= 12,\\
x &= 40.
\end{aligned}$$

CHECK. $$\frac{72 + 40}{120 + 40} = \frac{112}{160} = 0.7 = 70\%.$$

EXERCISES II. A

Solve for x:

1. $5x + 35 = 0.$
2. $2x - 7 = 3.$
3. $6x - 7 = 2x + 17.$
4. $\frac{1}{2}x - 6 = 3x + \frac{2}{3}.$
5. $x - 3(1 - x) = 5 - 2(3x + 2).$
6. $x^2 + 5x - 3 = (x + 5)(x - 3).$
7. $ax + b = cx + d.$
8. $(x + 7)(x - 3) = (x - 5)(x - 2).$
9. $(x - 5)(x - 4) = x^2 + 9x + 10.$
10. $\dfrac{3x + 2}{4} = \dfrac{x}{2} - \dfrac{5}{6}.$
11. Solve the following equation (a) for x, (b) for y:

$$4x - 3y = 6.$$

12. Solve the following equation (a) for x, (b) for y:

$$Ax + By + C = 0.$$

13. Solve the following equation (a) for x, (b) for k:

$$5kx - 3k + 7 = 0.$$

14. Solve $s = vt + s_0$ (a) for v, (b) for t.

15. Solve $R = \frac{Kl}{d^2}$ for l.

16. Solve the following equation (a) for w_1, (b) for w_2, (c) for w_3:

$$S = \frac{w_1}{w_1 + w_2 - w_3}.$$

17. One number is 3 less than another; their sum is 51. What are the numbers?

18. Find three consecutive numbers whose sum is 258.

19. Find five consecutive odd numbers whose sum is 255.

20. Find four consecutive even numbers whose sum is 156.

21. The length of a rectangle is 3 inches greater than its width. Its perimeter is 20 inches. What are its dimensions?

22. A man is twice as old as his son. Twelve years ago he was three times as old. Find their ages.

23. A man is five times as old as his son. In 7 years he will be three times as old as his son will be at that time. How old are they?

24. In a sum of money composed of dimes and quarters there are twice as many dimes as quarters. If the sum of money amounts to $32.85 how many quarters are there?

25. A sum of $2385 was raised for a benefit by selling 2000 tickets to a show. Some of the tickets were sold at $1.00 apiece, the rest at $1.50 apiece. How many of each kind were sold?

26. A man bought 50 stamps for $2.25. Some of them were ordinary 3-cent stamps, the others were 8-cent air mail stamps. How many of each did he buy?

27. A man paid $2.25 for 12 gallons of gasoline. Part of this was regular gasoline at 17 cents a gallon and part was ethyl at 20 cents a gallon. How much of each kind did he buy?

28. How much 90 per cent alcohol must be added to 20 gallons of a mixture which is 82 per cent alcohol to make the mixture 85 per cent alcohol?

29. How many pounds of coffee costing 25 cents a pound must be added to 60 pounds of coffee costing 30 cents a pound so that the mixture can be sold for 35 cents a pound at a gain of $7?

30. A dairyman has 1000 pounds of milk which contains $2\frac{1}{2}$ per cent butterfat. How many pounds of it must he remove, and replace with an equal amount of cream containing 25 per cent butterfat, to obtain a mixture which contains $3\frac{1}{2}$ per cent butterfat?

31. A man has \$8000 which he places at interest, part at 3 per cent, the rest at $3\frac{1}{2}$ per cent. The total annual interest which he receives from these two investments is \$251.75. How much is invested at each rate?

32. An investment of \$5000 yields a certain rate of interest; another investment of \$2000 yields a rate which is $\frac{1}{2}$ per cent higher. The annual yield from the two investments is \$255. Find the rates.

33. A man drove a certain distance at the rate of 50 miles an hour and returned at the rate of 40 miles an hour. The round trip required 2 hours and 15 minutes. Find the total distance that he drove.

34. Two airplanes set out simultaneously from airports 500 miles apart. They fly toward each other and meet at the end of 1 hour and 20 minutes. The speed of one plane is 15 miles per hour greater than that of the other. Find the speed of each.

35. An airplane flew in a direct line from its base at the rate of 200 miles per hour and returned over the same route at the rate of 180 miles per hour, requiring 15 minutes longer on the return trip. How far from its base did it fly?

36. An airplane climbed at an average vertical speed of 200 feet per minute and descended at once at an average vertical speed of $12\frac{1}{2}$ miles per hour. The flight was performed in 26 minutes. What altitude did the plane reach?

37. A man fires a gun at a target and 2 seconds later hears the sound of the bullet striking the target. Assuming that the bullet travels at a speed of 2500 feet per second and that sound travels at a speed of 1100 feet per second, find the distance to the target.

38. How soon after 12 o'clock are the hands of a clock together again?

SOLUTION. Let x = number of minute spaces minute hand travels. Then $x/12$ = number of minute spaces hour hand travels. Since the minute hand goes completely around the dial and overtakes the hour hand, it travels 60 more minute spaces than the hour hand, and

$$x - \frac{x}{12} = 60,$$

from which we get $x = 65\frac{5}{11}$. The answer is therefore 1 hr. $5\frac{5}{11}$ min.

39. How soon after 2 o'clock will the hands of a clock extend in opposite directions?

40. How soon after 3 o'clock will the hands of a clock again form a right angle?

41. It is now between 9 and 10 o'clock. In 4 minutes the hour hand of a clock will be directly opposite the position occupied by the minute hand 3 minutes ago. What time is it?

A **lever** is a rigid bar having a single point of support, called the **fulcrum.** If a weight w is attached to the lever at a distance d from the fulcrum then d is called the **lever arm** of the weight w, and the product of the weight and lever arm, wd, is called the **moment** of the force about the fulcrum. In physics it is shown that when two or more weights are attached to a lever, then, *if the lever balances, the sum of the moments on one side of the fulcrum is equal to the sum of the moments on the other side.* Thus, for the lever in Fig. 4 we have $w_1d_1 = w_2d_2$.

FIG. 4

Unless the contrary is stated, the weight of the lever itself will be disregarded. If the weight of the lever is to be taken into consideration, it will be assumed that it is distributed **uniformly** (i.e., each linear foot of it weighs the same amount). In such a case *the entire weight of any part of the lever may be regarded as concentrated at the mid-point of that part.*

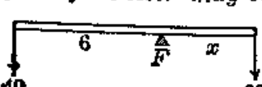


FIG. 5

42. Two children are balanced on a see-saw. The smaller, weighing 40 pounds, is 6 feet from the balancing point. How far from the balancing point must the other sit if he weighs 60 pounds?

SOLUTION. Let x = no. of ft. from balancing point.

Moment of 1st child $= 40 \cdot 6 = 240$.

Moment of 2nd child $= 60x$.

$$60x = 240,$$
$$x = 4.$$

43. A board 10 feet long weighs 8 pounds per foot. It is supported at a point 3 feet from one end. How large a weight must be attached at the extremity of the short end of the board to make it balance?

SOLUTION. *Method 1.* Let x = weight (in lb.). Short end of board weighs $3 \times 8 = 24$ lb. Since this may be considered as concentrated at its midpoint, its lever arm is 1.5 ft. and its moment is 24×1.5. Long end of board weighs $7 \times 8 = 56$ lb., its lever arm is 3.5 ft., and its moment is 56×3.5.

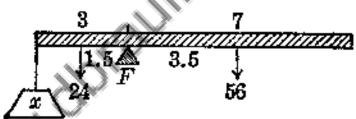


FIG. 6

Equating the sum of moments on the left of the balancing point to the moment on the right, we have

$$3x + 24 \times 1.5 = 56 \times 3.5,$$
$$x = 53\tfrac{1}{3} \text{ lb.}$$

Method 2. Weight of board is $10 \times 8 = 80$ lb. This may be regarded as concentrated at its midpoint, which is 2 ft. to right of fulcrum. Its moment is therefore $80 \times 2 = 160$, and we have

$$3x = 160, \quad x = 53\tfrac{1}{3}.$$

44. Two children weighing 60 and 40 pounds respectively are on opposite ends of a 12-foot board. Where must the point of support be placed if the board is to balance?

45. If, in the preceding exercise, the board weighs 5 pounds per foot, where must the point of support be placed?

46. A workman uses a crowbar 5 feet long. If the fulcrum is 9 inches from one end of the bar, how large a weight can a man move by exerting a force of 180 pounds?
47. If the fulcrum in the preceding exercise is 8 inches from one end of the bar, how much force must be exerted to move a weight of 975 pounds?
48. A crowbar is 5 feet long. Where must the fulcrum be placed so that a weight of 1225 pounds can be moved by exerting a force of 175 pounds?
49. A rod has a weight of 35 pounds attached at one end, a weight of 25 pounds at the other. It balances at a point 6 inches from the middle. How long is the rod?
50. A rod is 10 feet long. A weight of 75 pounds is attached to one end, a weight of 50 pounds to the other. At a distance of 2 feet from the 50-pound weight a third weight of 35 pounds is attached. Where is the balancing point (a) if the weight of the rod can be neglected? (b) if the rod weighs 4 pounds per foot?

15. Linear equations in two unknowns.

We shall recall, by means of an example, a method of solving a pair, or **system,** of linear equations in two unknowns (simultaneous equations).

Example.

Solve the equations

$$2x - 3y = 16,$$
$$5x + 2y = 2.$$

Solution. Multiply the first equation by 2 and the second by 3, in order to make the coefficients of y numerically equal. We have

$$4x - 6y = 32,$$
$$15x + 6y = 6.$$

Adding, we get $19x = 38.$

Dividing by 19 gives $x = 2.$

From the second of the original equations we find

$$2y = 2 - 5x = 2 - 10 = -8,$$

or

$$y = -4.$$

NOTE. It is usually better to obtain the value of the second unknown independently rather than by this method.

It is readily seen that the values $x = 2$, $y = -4$ satisfy both equations.

The equations may also be solved by solving one of them for y in terms of x, say, then substituting this expression in the other and solving the resulting equation for x.

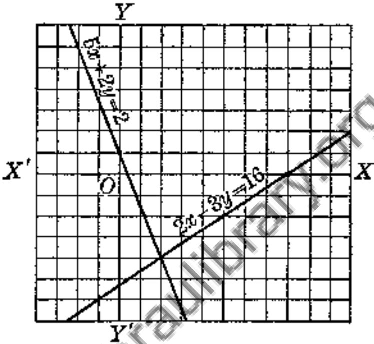


FIG. 7

In Fig. 7 the lines which are the graphs of the two equations are shown. It will be observed that they intersect at the point $x = 2$, $y = -4$, which is the solution of the two equations. Thus, the solution might have been obtained graphically by plotting the two lines and reading off the coordinates of the point at which they intersect. Ordinarily this method gives only an approximate result, whereas the algebraic process given above is exact.

Two equations such as

$$2x + y = 4,$$
$$2x + y = 8,$$

have no solution. They are said to be **inconsistent.** Their graphs (Fig. 8) are parallel lines and consequently do not intersect.

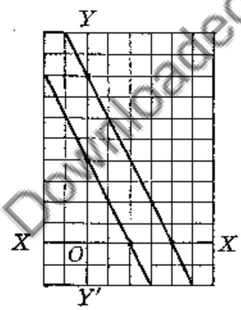


FIG. 8

Two equations such as

$$2x + y \ = 4,$$
$$4x + 2y = 8,$$

in which the second can be obtained by multiplying both sides of the first by 2, are said to be **dependent.** Any pair of values of x and y which satisfies one will satisfy the other. Their graphs coincide.

16. Linear equations in three or more unknowns.

To solve linear equations involving three or more unknowns, we take them in pairs and eliminate one of the unknowns, reducing the equations to a set in which there is one less equation and one less unknown. This process is repeated until we have a single equation with only one unknown, which can be solved.

Example.

Solve the simultaneous equations:

$$5x - y + 4z = 5, \quad (1)$$
$$2x + 3y + 5z = 2, \quad (2)$$
$$7x - 2y + 6z = 5. \quad (3)$$

SOLUTION. We shall first combine equations (1) and (2), eliminating y, and shall then combine (1) and (3), also eliminating y.

Multiply (1) by 3. This gives

$$15x - 3y + 12z = 15. \quad (4)$$

Add (2) and (4). Result:

$$17x + 17z = 17,$$

or

$$x + z = 1. \quad (5)$$

Multiply (1) by 2, getting

$$10x - 2y + 8z = 10. \quad (6)$$

Subtract (3) from (6). We get

$$3x + 2z = 5. \quad (7)$$

We are now ready to combine (5) and (7). Multiplying (5) by 2, we obtain

$$2x + 2z = 2, \tag{8}$$

and subtracting (8) from (7) we find

$$x = 3.$$

Substituting this value of x in (5), we get

$$z = -2.$$

Finally, by putting $x = 3$, $z = -2$ in (1) we obtain

$$y = 5x + 4z - 5 = 15 - 8 - 5 = 2.$$

By substituting $x = 3$, $y = 2$, $z = -2$, in the original equations (1), (2), (3), we verify the correctness of the solution.

(For a method of solving sets of linear equations by means of determinants see Chapter XVIII.)

EXERCISES II. B

Solve the following simultaneous equations for x and y:

1. $x + y = 25$, $x - y = 32$.

2. $3x + y = 21$, $4x + y = 23$.

3. $5x + 3y = 26$, $2x - 9y = 41$.

4. $4x - 5y = -16$, $6x - 7y = -21$.

5. $3x - 7y = 53$, $7x - 3y = -23$.

6. $21x + 22y = 69$, $15x - 4y = 0$.

7. $y = \frac{3}{5}x - 2$, $x = \frac{3}{8}y + \frac{17}{2}$.

8. $\frac{1}{2}y = \frac{2}{3}x + \frac{5}{6}$, $4x - 3y = -5$.

9. $y = \frac{2}{5}x - 7$, $4x - 10y = 25$.

10. $17x - 13y = 0$, $12x - 23y = 0$.

11. $13x + 27y = 303$, $22x + 23y = 127$.

12. $3x - 8y - 15 = 0$, $5x - 12y - 13 = 0$.

13. $6x - 7y = 9,$
$5x - 3y = 4.$

14. $4x + 7y = 4,$
$18x - 5y = 9.$

15. $\frac{1}{5}x + \frac{1}{4}y = 3\frac{1}{2},$
$\frac{1}{3}x + \frac{1}{2}y = 6.$

16. $0.3x + 1.7y = 36.4,$
$2.5x - 1.3y = -52.4.$

17. $4x + 5y = 48c,$
$3x - 2y = 13a.$

18. $3x + ay = 17,$
$7x - ay = 3.$

19. $\frac{3}{x} - \frac{2}{y} = \frac{13}{12},$
$\frac{5}{x} - \frac{6}{y} = \frac{9}{4}.$

20. $\frac{6}{x} + \frac{9}{y} = 21$
$\frac{8}{x} + \frac{15}{y} = 32.$

SUGGESTION. Solve first for $\frac{1}{x}$ and $\frac{1}{y}$.

21. $4ax - 3by = -3,$
$6ax + 5by = 43.$

22. $5ax - 2y = 37b,$
$2ax + 3y = -8b.$

23. $3ax + 5by = 21a,$
$5ax - 6by = -8b.$

24. $ax + by = m,$
$cx + dy = n.$

25. Solve exercises **21–22** for a and b.

Solve for x, y, and z:

26. $3x + 2y + z = 9,$
$4x - 3y + z = 22,$
$5x + 4y - z = 3.$

27. $6x + 2y - 3z = 15,$
$3x + 5y + 4z = -8,$
$4y + 5z = -13.$

28. $2x + 13y + 2z = -10,$
$3x - 7y - 4z = 69,$
$-5x + 2y - 3z = 1.$

29. $x + y = -1,$
$y + z = 9,$
$z + x = 4.$

30. $2x - 3y = 0,$
$4y - 5z = 2,$
$6z - 7x = -48.$

31. $13x - 12y + 7z = 77,$
$5x + 8y - 17z = -35,$
$9x - 11y - 10z = 29.$

32. $\frac{1}{x} + \frac{2}{y} = \frac{2}{3},$
$\frac{2}{y} + \frac{3}{z} = -\frac{5}{12},$
$\frac{3}{z} + \frac{4}{x} = \frac{7}{12}.$

33. $\frac{3}{x} + \frac{4}{y} + \frac{3}{z} = 3,$
$\frac{4}{x} + \frac{6}{y} - \frac{5}{z} = \frac{8}{3},$
$\frac{7}{x} + \frac{3}{y} - \frac{9}{z} = \frac{11}{4}.$

34. $\frac{1}{2}x + \frac{1}{3}y + \frac{1}{4}z = \frac{1}{2},$
$\frac{1}{5}x - \frac{1}{10}y + \frac{3}{8}z = -\frac{3}{2},$
$\frac{2}{3}x + \frac{5}{6}y - z = 0.$

35. $3x + 4y + 8z = 6,$
$9x - 5y + 6z = 13,$
$15x + 2y - 4z = 6.$

Solve for x, y, z, and t:

36. $2x - 3y + 4z - 5t = 0,$
$3x + 4y - 5z + 6t = 12,$
$5x - y - 11z - 4t = 11,$
$7x + 7y + 7z + 10t = 25.$

37. $x + y + z = 3,$
$y + z + t = 4,$
$z + t + x = 5,$
$t + x + y = 6.$

NOTE. Sometimes a special method will effect a quicker and neater solution. For example, exercise 37 can be solved as follows: Add the four equations and divide by 3. Subtract each equation in turn from the result.

38. $x + y = 1,$
$y + z = 2,$
$z + t = 3,$
$t + x = 2.$

39. $x + y + z - t = 1,$
$x + y - z + t = 2,$
$x - y + z + t = 3,$
$-x + y + z + t = 4.$

17. Worded problems.

Many worded problems lead to linear equations, as was seen in the section on linear equations in one unknown. Obviously there is no mechanical method which can be used for setting up the equations for all such problems, as there is no mechanical process which can be substituted for the process of thinking. Each problem, or rather each type of problem, must be thought out independently. The meaning of each statement must be clearly and definitely understood, and the various statements must then be translated into mathematical equations. However, for certain problems involving mixtures, rates of speed, or rates of interest, a tabular arrangement such as the one used in the following illustrative example may be of assistance in determining the equations which will lead to the solutions.

Example.

A grocer has two grades of coffee, selling at 40 cents and 50 cents a pound respectively. How much of each must he use to make a mixture of 100 pounds, to sell at 47 cents a pound?

SOLUTION.

	No. of lb. ×	Price per lb. =	Total price
1st Kind	x	40	$40x$
2nd Kind	y	50	$50y$
Mixture	100	47	4700

$$x + y = 100,$$
$$40x + 50y = 4700.$$

These equations have the solution

$$x = 30, \qquad y = 70,$$

which can be verified.

It will be noted that for any row in the table the product of the numbers in the first two columns gives the number in the third column, or that the number in the third column divided by the number in the first column or second column gives the number in the other column.

In problems involving rates of speed we have the fundamental relation *time* × *rate* = *distance*, and in problems in simple interest we know that *principal* × *rate* = *annual interest.* Consequently the above tabular arrangement can often be used to advantage in such problems

EXERCISES II. C

1. The sum of two numbers is 73, their difference is 41. What are the numbers?
2. A child's savings bank contains $5.05 in quarters and dimes. If there were three times as many quarters and half as many dimes the value of the contents would be $10.65. How many coins of each kind are there in the bank?
3. In 6 years a boy will be $\frac{3}{4}$ as old as his brother; 3 years ago he was $\frac{3}{5}$ as old as his brother. Find their ages.

4. A man wishes to place a fence around a rectangular lot. The fence costs 75 cents per foot for the sides and the back of the lot, and \$1.75 per foot for the front, making the total cost \$625.00. By using the cheaper kind of fence for the entire lot he could save \$100.00. Find the width and the depth of the lot.

5. A salesman receives 12 per cent commission on some articles which he sells for \$1.50 apiece and 15 per cent commission on some others which he sells for \$2.00 apiece. His commission was \$12.72 on a certain sale amounting to \$92.00. How many of each kind of article did he sell?

6. A man has two sums of money at interest, one at 3 per cent, the other at $2\frac{1}{2}$ per cent. His annual income from these two investments is \$195.00. On account of a drop in interest rates he finds that he can obtain only 2 per cent for his money. This decreases his annual income by \$45.00. How much does he have invested at each rate?

7. A man has two sums of money at interest, one at 3 per cent, one at $2\frac{1}{2}$ per cent. His annual income from these two investments is \$206.25. If he could obtain $\frac{1}{2}$ per cent more on each investment his annual income would be increased by \$40.00. How much does he have at interest at each rate?

8. A man has three sums of money invested, one at 2 per cent, one at 3 per cent, and one at $3\frac{1}{2}$ per cent. His total annual income from the three investments is \$246. The first of these yields \$14 per year more than the other two combined. If all of the money were invested at $2\frac{1}{2}$ per cent he would receive \$4 per year more than he does now. How much is invested at each rate?

9. An airplane made a trip of 200 miles against the wind in 1 hour. Returning with the wind it took only 50 minutes. Find the speed of the plane in calm air and the rate of the wind.

10. A motorist can drive from A to B in 2 hours and 24 minutes. By increasing his speed 10 miles per hour he can cut the time down to 2 hours. How far is it from A to B?

11. An army officer made the first part of a trip on a plane which flew at the rate of 210 miles per hour. At the landing field he was met by a jeep which took him the rest of the way to his

destination at the rate of 40 miles per hour. The trip required 3 hours and 15 minutes. On his return trip the jeep traveled at the rate of 50 miles per hour and the plane which he took flew at the rate of 200 miles per hour. The return journey required the same amount of time, but this included 9 minutes which he spent waiting for the plane to take off. Find the total distance that he flew and the total distance that he traveled by jeep.

12. A train is moving along a straight stretch of track parallel to a highway. An automobile traveling 60 miles per hour in the same direction as the train can pass it in 40 seconds. An automobile traveling 60 miles per hour in the opposite direction requires only 10 seconds to pass it. How long is the train and how fast is it moving?

13. A train 250 feet long passed another 998 feet long, traveling in the same direction on a parallel track, in 16 seconds. If the trains had been traveling in opposite directions they would have passed each other in half the time. Find the speed of each.

14. How much tin and how much iron must be added to 50 pounds of an alloy containing 10 per cent tin and 25 per cent iron to obtain an alloy containing 25 per cent tin and 50 per cent iron?

15. A grocer had three grades of coffee which cost him 12 cents a pound, 15 cents a pound, and 20 cents a pound respectively. He blended 100 pounds which he sold at 22 cents a pound, this being \$7.60 above cost. If he made half of the blend out of the cheapest grade, how many pounds of each of the other grades did he use?

16. A board 8 feet long is supported at a point 1 foot from the center. A weight of x pounds is attached at the end of the 3-foot lever arm, a weight of y pounds is attached at the end of the 5-foot lever arm. In order to make the board balance it is necessary to attach a 33-pound weight at a distance of 1 foot from the weight of x pounds. If the weights x and y are interchanged it is necesssary to place a 6-pound weight with the y-pound weight to effect a balance. Find x and y (a) neglecting the weight of the board, (b) assuming that the board weighs k pounds per foot.

17. A man has 72 Savings Bonds of total face value \$2475.

The denominations of the bonds are \$25, \$50, and \$100 respectively. He has six times as many \$50 bonds as he has \$100 bonds. How many of each kind has he?

18. Two pipes running simultaneously can fill a tank in 3 hours and 20 minutes. If both pipes run for 2 hours and the first is then shut off, it requires 2 hours more for the second to fill the tank. How long does it take each pipe to fill it alone?

SUGGESTION. Let x = no. of hr. for 1st pipe to fill tank. Then $1/x$ is part of tank that 1st pipe can fill in 1 hr. Use y similarly for 2nd pipe. Solve first for $1/x$ and $1/y$.

19. Three observation planes, A, B, and C, working together, can map a certain region in 4 hours. Planes A and B can map the region in 6 hours, planes B and C can map it in 6 hours and 40 minutes. How long would it take each of the planes working alone to map this region?

20. Observation planes A, B, and C, working together, can map a certain region in 4 hours. After they had worked for 3 hours B developed engine trouble, as a consequence of which A and C had to finish the job. This required 2 hours more. It turned out, however, that one-third of the photographs taken by B were spoiled. Meanwhile this plane had been repaired and was sent up with plane C on the following day. They took 45 minutes to photograph that part of the region for which B's photographs were spoiled. How long would it take each plane working alone to map the region?

21. The sum of the digits of a two-place number is 9. If the digits are reversed the number is decreased by 45. Find the number.

SUGGESTION. Let x = digit in tens' place, y = digit in units' place. Then the number is $10x + y$.

22. The sum of the digits in a three-place number is 13. If the digits are reversed and the resulting number is added to the original number the sum is 827; if the resulting number is subtracted from the original number the difference is 297. What is the number?

CHAPTER III

Factoring

18. Integral rational expressions.

An **integral rational expression, or polynomial,** in certain variables, for example, x, y, z, is a sum of terms of the type $kx^ay^bz^c$, in which k is a constant and the exponents a, b, c are positive whole numbers. The expressions

$$6x^3 + 5x^2 - 2x + 7,$$
$$4x^3y^2 - 3y^4 - x,$$
$$x^2y\sqrt{2} - \frac{2}{3}z$$

are integral rational expressions in the variables involved. The expressions

$$x^3\sqrt{2} + x\sqrt{y}, \qquad x^3 + \frac{3x^2}{y^3}$$

are integral rational expressions in x but not in y. (See section 8.)

The **degree** *of a term* of an integral rational expression is the sum of the exponents of the variables in the term. Thus, $4x^3y^2$ is of degree 5 in x and y, although it is of degree 3 in x alone and of degree 2 in y alone. The degree of an integral rational expression is that of its term or terms of maximum degree.

19. Factoring.

By **factoring** an integral rational expression is meant the process of finding two or more integral rational expressions whose product is the given expression. Unless otherwise specified, we shall limit the process to factoring integral rational expressions whose coefficients are rational * numbers, into factors of the same type. Such an expression will be called **prime** if it has no other factors of this type besides itself or its negative, and 1 (or, of course, c times itself and $1/c$, where c is a rational constant different from 0).

In the following sections will be given some of the types of factoring which are of most frequent occurrence.

20. Elementary factor types.

Common monomial factor:

$$ax + ay = a(x + y). \tag{1}$$

Example.

$$5x^3 - 10x^2z = 5x^2(x - 2z).$$

Difference of two squares:

$$a^2 - b^2 = (a + b)(a - b). \tag{2}$$

Example.

$$16x^2 - 25y^2 = (4x + 5y)(4x - 5y).$$

Square of a binomial:

$$a^2 + 2ab + b^2 = (a + b)^2, \tag{3}$$
$$a^2 - 2ab + b^2 = (a - b)^2. \tag{4}$$

For a trinomial to be a perfect square, one term must be twice the product of the square roots of the other two.

* A **rational number** is a number which can be expressed exactly as the ratio of two whole numbers; an **irrational number** is one which cannot be so expressed. Thus, 2/3, −17/6, 5, and 0 are rational numbers; $\sqrt{2}$ and π are irrational numbers.

Example.

$$16x^2 - 24x + 9 = (4x - 3)^2.$$

Trinomial of the form:

$$x^2 + (a + b)x + ab = (x + a)(x + b). \qquad (5)$$

Examples.

$$x^2 + 5x + 6 = (x + 3)(x + 2),$$
$$x^2 - 5x + 6 = (x - 3)(x - 2),$$
$$x^2 + 5x - 6 = (x + 6)(x - 1),$$
$$x^2 - 5x - 6 = (x - 6)(x + 1).$$

It will be noted that in factoring this type we must find two numbers whose product is the constant term and whose algebraic sum is the coefficient of the first-degree term.

If the constant term is positive, these two numbers will have the same sign, namely, that of the first-degree term.

If the constant term is negative, the numbers will have unlike signs, and the numerically greater will have the sign of the first-degree term.

Trinomial of the form:

$$acx^2 + (ad + bc)x + bd = (ax + b)(cx + d). \qquad (6)$$

This is like the preceding type, but is usually more difficult to factor.

Example.

$$6x^2 + 7x - 20 = (2x + 5)(3x - 4).$$

In factoring this type by inspection one can write down various possibilities until he discovers the right combination. For instance, in the present example he might try several combinations, such as $(6x - 1)(x + 20)$, before discovering the correct one.

(In Chapter VI it will be shown how types (5) and (6) can always be factored by solving a quadratic equation.)

Grouping:

$$ax + ay + bx + by = a(x + y) + b(x + y)$$
$$= (a + b)(x + y). \qquad (7)$$

Example.

$$3k^2 - 6mk - k + 2m = 3k(k - 2m) - (k - 2m)$$
$$= (3k - 1)(k - 2m).$$

Type (6) above may also be factored by grouping.

Example.

$$6x^2 + 7x - 20.$$

Multiply together the constant term and the coefficient of x^2.

$$6(-20) = -120.$$

Find two factors of -120 whose algebraic sum is equal to the coefficient of x, viz., 7. These factors are 15 and -8. Then,

$$6x^2 + 7x - 20 = 6x^2 + 15x - 8x - 20$$
$$= 3x(2x + 5) - 4(2x + 5)$$
$$= (3x - 4)(2x + 5).$$

21. More complicated types.

Sum of two cubes:

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2). \qquad (1)$$

Difference of two cubes:

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2). \qquad (2)$$

Cube of a binomial:

$$a^3 + 3a^2b + 3ab^2 + b^3 = (a + b)^3, \tag{3}$$

$$a^3 - 3a^2b + 3ab^2 - b^3 = (a - b)^3. \tag{4}$$

Sum of two odd powers: The sum of two odd powers is always divisible by the sum of the numbers.

Example.

$$a^5 + b^5 = (a + b)(a^4 - a^3b + a^2b^2 - ab^3 + b^4). \tag{5}$$

Note the form of the second factor: The coefficients of the various terms are alternately +1 and −1; the exponent of a decreases by 1 from term to term, while the exponent of b increases by 1 from term to term (b first appears in the second term).

Difference of two odd powers: The difference of two odd powers is always divisible by the difference of the numbers.

Example.

$$a^5 - b^5 = (a - b)(a^4 + a^3b + a^2b^2 + ab^3 + b^4). \tag{6}$$

The *difference of two even powers* should be factored as the *difference of squares.*

Example.

$$a^6 - b^6 = (a^3)^2 - (b^3)^2 = (a^3 + b^3)(a^3 - b^3)$$
$$= (a + b)(a^2 - ab + b^2)(a - b)(a^2 + ab + b^2).$$

Each factor should be refactored if possible until the expression is reduced to prime factors.

NOTE. If we had not limited the type of factors to be considered to integral rational expressions with rational coefficients (section 19), $x^2 - 2y^2$ could be factored into $(x + y\sqrt{2})(x - y\sqrt{2})$, and $x - y$ could be factored into $(\sqrt{x} + \sqrt{y})(\sqrt{x} - \sqrt{y})$.

The sum of two even powers cannot be factored *as such*. Note, however, that

$$\begin{aligned} x^6 + y^6 &= (x^2)^3 + (y^2)^3 \\ &= (x^2 + y^2)[(x^2)^2 - x^2y^2 + (y^2)^2] \\ &= (x^2 + y^2)(x^4 - x^2y^2 + y^4). \end{aligned}$$

See also example 2 in next section.

22. Other types.

Frequently an expression can be reduced to the difference of two squares.

Example 1.

$$\begin{aligned} x^2 - y^2 - z^2 + 2yz &= x^2 - (y^2 - 2yz + z^2) \\ &= x^2 - (y - z)^2 \\ &= [x + (y - z)][x - (y - z)] \\ &= (x + y - z)(x - y + z). \end{aligned}$$

Example 2.

$$x^4 + 4y^4.$$

Add and subtract $4x^2y^2$:

$$\begin{aligned} x^4 + 4y^4 &= x^4 + 4x^2y^2 + 4y^4 - 4x^2y^2 \\ &= (x^2 + 2y^2)^2 - (2xy)^2 \\ &= (x^2 + 2y^2 + 2xy)(x^2 + 2y^2 - 2xy). \end{aligned}$$

Example 3.

$$9x^4 - 6x^2y^2 + 25y^4.$$

This would be a perfect square if the middle term were $\pm 30x^2y^2$. If we add and subtract $36x^2y^2$ we get

$$\begin{aligned} 9x^4 - 6x^2y^2 + 25y^4 &= 9x^4 + 30x^2y^2 + 25y^4 - 36x^2y^2 \\ &= (3x^2 + 5y^2)^2 - (6xy)^2 \\ &= (3x^2 + 5y^2 + 6xy)(3x^2 + 5y^2 - 6xy). \end{aligned}$$

It should be remembered that

$$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc, \quad (7)$$

and that in general *the square of a polynomial is the sum of the squares of its terms plus twice the product of each term multiplied separately by every term that follows it.* Remembering this sometimes enables us to recognize a lengthy expression as the square of a polynomial.

Example.

$$9x^2 + 6xy + y^2 - 12xz - 4yz + 4z^2 = (3x + y - 2z)^2.$$

Sometimes a polynomial such as

$$x^3 - 2x^2 - 2x - 3,$$

in which the coefficients are all whole numbers, that of the term of highest degree being 1, can be factored by inspection. Possible factors are $x \pm k$, where k is a factor of the constant term. In this example the possible factors are $x + 1$, $x - 1$, $x + 3$, $x - 3$, the last of which is found, by division, to be a factor. This method will be discussed further in Chapter XIII.

23. Suggestions for factoring.

The secret of factoring is the ability to recognize type forms. The following suggestions may prove helpful:

First *remove monomial factors* if any exist. It may then be possible to classify the expression under one of the following heads:

Binomial:

Difference of squares.
Difference of cubes.
Sum of cubes.

Difference of odd powers.
Sum of odd powers.
Reducible to difference of squares by adding and subtracting a perfect square.

Trinomial:

Square of binomial.
Type $x^2 + (a + b)x + ab$.
Type $acx^2 + (ad + bc)x + bd$.
Reducible to difference of squares by adding and subtracting a perfect square.

Polynomial of four or more terms:

Capable of being grouped.
Reducible to difference of squares by rearrangement.
Cube of binomial.
Square of polynomial.
Divisible by $x \pm k$, where k is a factor of the constant term.

Always see whether any factor can be factored further.

A check is afforded by multiplying together the factors obtained. The product must equal the original expression if the work is correct.

EXERCISES III. A

Factor completely:

1. $4axy - 6ayz$.
2. $14ab^2 - 49a^2b^3c - 21ab^4c^2$.
3. $16x^2 - 25y^2$.
4. $81a^2b^4 - 16$.
5. $18x^4 - 8y^2$.
6. $75a^4 - 147a^2b^2$.
7. $9a^2 - 24ab + 16b^2$.
8. $64x^4 + 80x^2yz + 25y^2z^2$.
9. $98a^2 - 112ax + 32x^2$.
10. $x^2 - 8x + 15$.
11. $x^2 - 4x - 12$.
12. $x^3 + 5x^2y - 24xy^2$.
13. $10 + 3x - x^2$.
14. $12 - x - x^2$.
15. $2x^2 - x - 6$.
16. $6x^2 + 11x - 10$.
17. $12x^2 + 35x + 18$.
18. $18x^2 + x - 4$.

19. $18x^2 - 39xy + 20y^2$.
20. $24a^2 + 11ab - 18b^2$.
21. $3ax + 5ay + 6x + 10y$.
22. $12x^2 - 12ay + 16xy - 9ax$.
23. $a^3 - a^2 + a - 1$.
24. $a^2 - b^2 + a + b$.
25. $a^3 + b^3 + a + b$.
26. $a^2 + b^2 - c^2 - 2ab$.
27. $2xz - x^2 + y^2 - z^2$.
28. $9x^2 + 4y^2 - 4z^2 - 12xy$.
29. $9a^2 + 6ab + 2xy + 4b^2 - x^2 - y^2$.
30. $8ad + 4a^2 + 4ab + 4bd - 4ac - 4cd - 2bc + b^2 + c^2 + 4d^2$.
31. $8a^3 - b^3 - 12a^2b + 6ab^2$.
32. $a^3 - 8$.
33. $x^6 + 8$.
34. $81x^4 - y^4$.
35. $a^6 + b^6$.
36. $x^5 + 32$.
37. $3x^6 - 12y^6$.
38. $a^7 + b^7$.
39. $a^8 - b^8$.
40. $125x^3 + 27y^3$.
41. $a^6 + b^{12}$.
42. $a^6 - b^{12}$.
43. $8a^4 + 2b^4$.
44. $16x^4 + 23x^2y^2 + 9y^4$.
45. $9x^4 - 21x^2y^2 + 4y^4$.
46. $x^4 - 10x^2y^2 + 9y^4$.
47. $25a^4 + 45a^2b^2 + 49b^4$.
48. $64a^8 + 7a^4b^2 + 4b^4$.
49. $x^4 - 10x^2y^2 + 9y^4$.
50. $x^4 + 4x^2y^2 - 12y^4$.
51. $6x^4 - x^2y^2 - 15y^4$.
52. $x^9 - y^9$.
53. $x^3 - 4x + 3$.
54. $x^3 - 4x^2 + x + 6$.
55. $a^{b+2} - a^bc^2$.
56. $a^{10} + 1$.

Evaluate the following expressions by first factoring them:

57. $(68)^2 - (67)^2$.
58. $(99)^2 - (98)^2$.
59. $(999)^2 - (998)^2$.
60. $(55)^2 - (45)^2$.

24. Highest common factor and lowest common multiple.

The **highest common factor** (H. C. F.) of two or more expressions is the expression of highest degree that can be divided into all of them exactly.

The **lowest common multiple** (L. C. M.) of two or more expressions is the expression of lowest degree that is exactly divisible by all of them.

To obtain the H. C. F. and the L. C. M. of two or more expressions, separate each expression into its prime factors. The H. C. F. is the product formed by taking each factor to the lowest power to which it appears in any expression, non-appearance being called degree zero. The L. C. M.

is the product formed by taking each factor to the highest power to which it appears in any expression.

Example 1.

Find the H. C. F. and the L. C. M. of $2x^2y^3$, $4xy^5$, $18x^7y^2z$.

SOLUTION. Rewrite in the respective forms

$$2x^2y^3,\ 2^2xy^5,\ 2\cdot 3^2x^7y^2z.$$

$$\text{H. C. F.} = 2xy^2.$$

$$\text{L. C. M.} = 2^2 3^2 x^7y^5z = 36x^7y^5z$$

Example 2.

Find the H. C. F. and the L. C. M. of

$$x^2 + 2xy + y^2,\ x^2 - y^2,\ x^2 - xy - 2y^2.$$

SOLUTION.

$$x^2 + 2xy + y^2 = (x + y)^2,$$

$$x^2 - y^2 = (x + y)(x - y),$$

$$x^2 - xy - 2y^2 = (x + y)(x - 2y).$$

$$\text{H. C. F.} = x + y.$$

$$\text{L. C. M.} = (x + y)^2(x - y)(x - 2y).$$

EXERCISES III. B

1. Prove that the product of the highest common factor and the lowest common multiple of *two* expressions is equal to the product of the two expressions.

Find the highest common factor and the lowest common multiple of the following expressions:

2. $72x^4y^3z^2$, $324xy^3z^4$.
3. $x^2 - y^2$, $x^2 + 2xy + y^2$.
4. $x^2 - 5x + 6$, $x^2 + 3x - 18$.
5. $6x^2 - 11x - 10$, $12x^2 - x - 6$.
6. $81a^4 - 16b^4$, $9a^2 - 4b^2$.
7. $81a^4 + 16b^4$, $9a^2 + 4b^2$.

8. $50x^2 - 18y^2,\ 15x + 9y.$
9. $1125a^5b^4c^2,\ 16875ab^4c^7,\ 2025a^3b^6c^3.$
10. $x^2 + 2xy + y^2,\ x^3 + y^3,\ x^2 - y^2.$
11. $x^3 - x^2y + xy^2,\ x^3 + y^3.$
12. $x^2 - x - 6,\ x^2 + x - 12,\ x^2 - 9x + 18.$
13. $a^4 + 4b^4,\ a^4 - 2a^3b + 2a^2b^2.$
14. $x^6 - y^6,\ x^4 - y^4.$
15. $x^6 + y^6,\ x^4 - y^4.$
16. $15x^2 - 8x - 12,\ 20x^2 - 39x + 18,\ 10x^2 + 13x - 30.$

CHAPTER IV

Fractions

25. Elementary principles.

The value of a fraction is unchanged if the numerator and the denominator are both multiplied or both divided by the same number, zero excepted, since (section 28) this is equivalent to multiplying the fraction by 1.

This principle can be used in reducing a fraction to its lowest terms.

Example.

Reduce $\dfrac{4x^2y^7z}{6x^5y^3}$ to lowest terms.

Solution. Divide numerator and denominator by $2x^2y^3$, getting

$$\frac{2y^4z}{3x^3}.$$

A fraction may be regarded as having three signs associated with it: the sign of the numerator, the sign of the denominator, and the sign preceding the fraction. *Any two of these signs may be changed without changing the value of the fraction.*

Example.

$$\frac{a}{b} = -\frac{-a}{b} = -\frac{a}{-b} = \frac{-a}{-b}.$$

26. Addition and subtraction of fractions.

To add or subtract fractions we must reduce them to a common denominator. It is usually best to reduce them to their **lowest common denominator** (L. C. D.), that is, the lowest common multiple of their denominators.

Example.

$$\frac{3}{4x} + \frac{5}{6x^2} = \frac{9x + 10}{12x^2}.$$

The dividing line of a fraction performs the same service as signs of grouping placed around the numerator.

Example.

$$\frac{3}{4x} - \frac{2x - 5}{6x^2} = \frac{3x \cdot 3 - 2(2x - 5)}{12x^2} = \frac{9x - 4x + 10}{12x^2} = \frac{5x + 10}{12x^2}.$$

27. Mixed expressions.

A fraction can often be reduced to a **mixed expression** (that is, the sum of a fraction and an expression free from fractions) just as in arithmetic an improper fraction is reduced to a mixed number (e.g., $\frac{23}{5} = 4\frac{3}{5}$).

Example.

Reduce $\dfrac{2x^2 + 3x + 5}{x + 1}$ to a mixed expression.

Solution. Divide $2x^2 + 3x + 5$ by $x + 1$.

$$\begin{array}{r l}
 & 2x + 1 \quad (= \text{quotient}) \\
x + 1 \big) & 2x^2 + 3x + 5 \\
 & \underline{2x^2 + 2x} \\
 & \qquad x + 5 \\
 & \qquad \underline{x + 1} \\
 & \qquad\quad 4 \ (= \text{remainder})
\end{array}$$

$$\frac{2x^2 + 3x + 5}{x + 1} = 2x + 1 + \frac{4}{x + 1}.$$

EXERCISES IV. A

Reduce to lowest terms:

1. $\dfrac{36xy^3z^9}{54x^8y^2z^3}$.

2. $\dfrac{a^2 - 2ab + b^2}{a^2 - b^2}$.

3. $\dfrac{x^2 - 7x + 12}{x^3 + 3x - 18}$.

4. $\dfrac{x^2 - 7xy - 8y^3}{x^3 + y^3}$.

5. $\dfrac{6x^2 + x - 12}{2x^2 - x - 6}$.

6. $\dfrac{(a - b)(x + y)}{(b - a)(x - y)}$.

Combine into a single fraction:

7. $\dfrac{7}{18} + \dfrac{5}{12}$.

8. $\dfrac{3}{8a^2} + \dfrac{5}{6a}$.

9. $\dfrac{3}{x - 2y} + \dfrac{2}{x + 2y}$.

10. $\dfrac{3}{x - 2y} - \dfrac{2}{x + 2y}$.

11. $\dfrac{5}{x - 3} - \dfrac{3}{x - 5}$.

12. $\dfrac{3}{2ab} - \dfrac{4}{3bc} + \dfrac{5}{4ac}$.

13. $\dfrac{1}{a^2 - b^2} - \dfrac{1}{a^2 + 2ab + b^2}$.

14. $\dfrac{a}{a^2 - 2ab + b^2} - \dfrac{b}{a^2 - b^2}$.

15. $\dfrac{3}{a - b} - \dfrac{2}{b - a}$.

SOLUTION. Change the sign of the denominator of the second fraction and change the sign preceding it. Then we have

$$\frac{3}{a - b} + \frac{2}{a - b} = \frac{5}{a - b}.$$

16. $\dfrac{x}{x - 5} - \dfrac{5}{5 - x}$.

17. $\dfrac{3a}{3a - 2b} + \dfrac{2a}{2b - 3a}$.

18. $\dfrac{5}{(x - 2)(x + 3)} + \dfrac{6}{(x + 3)(x - 4)}$.

19. $\dfrac{1}{(x - y)(y - z)} + \dfrac{2}{(y - z)(z - x)} + \dfrac{3}{(z - x)(x - y)}$.

20. $\frac{2}{x-2} - \frac{3}{x+3} + \frac{4}{x-4}$.

21. $\frac{1}{(a-b)^2} + \frac{a}{a-b} - 1$.

22. $\frac{a^2}{(a-b)^2} - \frac{b}{a-b} - \frac{a}{b-a}$.

23. $2a + 3b + \frac{12b^2}{a-4b}$.

24. $\frac{1}{x^2+x-6} - \frac{1}{2x^2-x-6} + \frac{1}{2x^2+5x+3}$.

Reduce the following fractions to mixed expressions:

25. $\frac{5x^2-6x+12}{x-4}$.

26. $\frac{6x^2-5x-7}{2x+1}$.

27. $\frac{6x^2-8x+1}{3x-2}$.

28. $\frac{x^3-5x^2-2x+3}{x^2-3x-4}$.

29. $\frac{x^3+4x^2-5x-2}{x^2+1}$.

30. $\frac{2x^3-20x+75}{x^2+5x-7}$.

28. Multiplication of fractions.

To multiply fractions we multiply their numerators to obtain the numerator of the product, and multiply their denominators to obtain the denominator of the product. Factors common to numerator and denominator should be divided out.

Examples.

$$\frac{x^3}{y^3} \cdot \frac{y}{ax} = \frac{x^2}{ay^2},$$

$$\frac{(a+b)^2}{a^2+b^2} \cdot \frac{ab}{a^2-b^2} = \frac{(a+b)^2}{a^2+b^2} \cdot \frac{ab}{(a+b)(a-b)} = \frac{ab(a+b)}{(a^2+b^2)(a-b)}.$$

29. Powers and roots of fractions.

A power of a fraction is the power of the numerator divided by the power of the denominator; a root of a fraction is the root of the numerator divided by the root of the denominator.

Examples.

$$\left(\frac{a}{b}\right)^2 = \frac{a^2}{b^2}, \qquad \sqrt{\frac{a^2}{9}} = \frac{a}{3} \quad (a > 0).$$

30. Division of fractions.

To divide one fraction by another, invert the divisor and multiply.

Example.

$$\frac{x}{y} \div \frac{a}{b} = \frac{x}{y} \cdot \frac{b}{a} = \frac{bx}{ay}.$$

EXERCISES IV. B

Perform the indicated operations and simplify:

1. $\frac{3x^4y^3z^5}{4a^2bc^7} \cdot \frac{6ab^2c^3}{5x^3y^3z^3}.$

2. $\frac{a^2+4a-12}{a^2+4} \cdot \frac{a^2-4}{a^2-4a+4}.$

3. $\frac{x^2-9}{x-6} \cdot \frac{x^2-5x-6}{x^2-x-6}.$

4. $\frac{a^3+b^3}{(a+b)^3} \cdot \frac{a^2+2ab+b^2}{a^2-ab+b^2}.$

5. $\frac{9a^2x^3z^5}{4b^7cy^4} \div \frac{12a^5b^5x^5}{7cy^3z^7}.$

6. $\frac{(x-y)^2}{x^2-y^2} \div \frac{x^2+y^2}{(x+y)^2}.$

7. $\frac{x^3+y^3}{x^2-y^2} \div \frac{x+y}{x^4-y^4}.$

8. $\frac{b-a}{(x-y)^2} \div \frac{a^2-2ab+b^2}{x^2-y^2}.$

9. $\left(\frac{xy^2z^3}{a^2b^2c^2} \div \frac{xyz}{a^3b^3c^3}\right)\frac{a^3b^4c^5}{x^3y^2z}.$

10. $\frac{xy^2z^3}{a^2b^2c^2} \div \left(\frac{xyz}{a^3b^3c^3} \cdot \frac{a^3b^4c^5}{x^3y^2z}\right).$

11. $\left(\frac{2rst}{3u^2v^2} \div \frac{5rs^2t^3}{6u^3v^4}\right) \div \frac{8uv}{9r^5s^4t^3}.$

12. $\frac{2rst}{3u^2v^2} \div \left(\frac{5rs^2t^3}{6u^3v^4} \div \frac{8uv}{9r^5s^4t^3}\right).$

13. $(x^3+y^3) \div \left(x - y + \frac{y^2}{x}\right).$

14. $\left(1 - \frac{y}{x}\right) \div \left(1 - \frac{x}{y}\right).$

15. $\left(1 - \frac{2y}{x} + \frac{y^2}{x^2}\right) \div \frac{x^2-y^2}{x^2y^2}.$

16. $\left(1 - \frac{x^2}{y^2}\right) \div \left(1 - \frac{y^3}{x^3}\right).$

31. Complex fractions.

A **complex fraction** is one which has one or more fractions in its numerator or in its denominator or in both. The methods of simplifying complex fractions will be illustrated by the following examples:

Example 1.

Simplify

$$\frac{\dfrac{x}{x-1}+\dfrac{1}{x+1}}{\dfrac{x}{x-1}-\dfrac{1}{x+1}}$$

SOLUTION. Multiply numerator and denominator by the L. C. D. of the fractions which occur in the numerator and denominator of the complex fraction. This L. C. D. is seen to be $(x+1)(x-1)$. The original expression becomes

$$\frac{(x+1)(x-1)\dfrac{x}{x-1}+(x+1)(x-1)\dfrac{1}{x+1}}{(x+1)(x-1)\dfrac{x}{x-1}-(x+1)(x-1)\dfrac{1}{x+1}}$$

$$=\frac{(x+1)x+(x-1)}{(x+1)x-(x-1)}=\frac{x^2+x+x-1}{x^2+x-x+1}$$

$$=\frac{x^2+2x-1}{x^2+1}.$$

Example 2.

Simplify

$$\frac{1}{x-\dfrac{1}{2x-\dfrac{x+1}{x}}}. \qquad (1)$$

SOLUTION. Consider the fraction

$$-\frac{1}{2x - \dfrac{x+1}{x}} \qquad (2)$$

which is found in the denominator. The L. C. D. of the fractions occurring in the numerator and denominator of (2) is x (since there is only one such fraction and its denominator is x). Multiply numerator and denominator of (2) by x. Then (2) becomes

$$-\frac{x}{2x^2 - x - 1},$$

and (1) takes the form

$$\frac{1}{x - \dfrac{x}{2x^2 - x - 1}}. \qquad (3)$$

The L. C. D. of the fractions occurring in numerator and denominator of (3) is $2x^2 - x - 1$. Multiplying numerator and denominator by this gives

$$\frac{2x^2 - x - 1}{2x^3 - x^2 - x - x} = \frac{2x^2 - x - 1}{2x^3 - x^2 - 2x}.$$

EXERCISES IV. C

Simplify:

1. $\dfrac{\dfrac{5}{6} - \dfrac{2}{3}}{\dfrac{3}{4}}$

2. $\dfrac{\dfrac{x}{a} + \dfrac{y}{a^3}}{\dfrac{x}{a^2}}$.

3. $\dfrac{\dfrac{x+1}{x}}{x + \dfrac{1}{x}}$.

4. $\dfrac{x + \dfrac{1}{x} - 2}{x - \dfrac{1}{x}}$.

5. $\dfrac{\dfrac{x^2}{y}+\dfrac{y^2}{x}}{\dfrac{x}{y}+\dfrac{y}{x}-1}.$

6. $\dfrac{\dfrac{a}{a-b}-\dfrac{b}{a+b}}{\dfrac{ab}{a^2-b^2}}.$

7. $\dfrac{\dfrac{1}{3x+7}+\dfrac{1}{3x-7}}{\dfrac{1}{9-\dfrac{49}{x^2}}}.$

8. $\dfrac{\dfrac{x}{x+y}+\dfrac{y}{x-y}}{\dfrac{x^2-y^2}{x^2+y^2}-\dfrac{x^2+y^2}{x^2-y^2}}.$

9. $\dfrac{1}{1-\dfrac{1}{1-x}}.$

10. $\dfrac{x-2}{x-2-\dfrac{x}{x-\dfrac{x-1}{x-2}}}.$

11. $\dfrac{1}{1-\dfrac{1}{2-\dfrac{1}{3-x}}}.$

12. $\dfrac{1}{2-\dfrac{3}{4+\dfrac{5}{6-x}}}.$

13. $1-\dfrac{1}{1-\dfrac{1}{1-\dfrac{1}{1-x}}}.$

14. $\dfrac{a-b}{a-b-\dfrac{1}{a+b+\dfrac{1}{a-b}}}.$

15. $\dfrac{\dfrac{x-1}{x-2}-\dfrac{x-2}{x-3}}{\dfrac{x-3}{x-4}-\dfrac{x-4}{x-5}}\div\dfrac{\dfrac{x-2}{x-1}-\dfrac{x-3}{x-2}}{\dfrac{x-4}{x-3}-\dfrac{x-5}{x-4}}.$

16. $\dfrac{\dfrac{x}{x^2-y^2}-\dfrac{x+y}{x^2+y^2}}{\dfrac{y}{x^2-y^2}+\dfrac{y}{x^2+y^2}}+\dfrac{\dfrac{x^2+y^2}{x^2+xy}}{\dfrac{x}{x-y}-\dfrac{y}{x+y}}.$

32. Equations involving fractions.

Before proceeding to the solution of equations involving fractions, it will be well to consider what algebraic operations are, in general, permissible in solving equations.

Two equations are **equivalent** if each possesses exactly the same roots as the other.

The following operations, when performed on an equation, always lead to an equivalent equation:

Adding the same number or expression to, or subtracting the same number or expression from, both sides.

Multiplying or dividing both sides by the same non-zero number or by the same expression, provided that the expression does not contain an unknown.

These operations, and no others, were used in solving linear equations in Chapter II.

The following operations, when performed on an equation, may lead to an equation containing ***extraneous*** *roots, that is, roots which are not roots of the original equation:*

Multiplying both sides by an expression containing an unknown.

Raising both sides to the same integral power.

It is therefore absolutely necessary to check the values obtained in solving an equation when either of these operations has been performed. For even if the work has been correct, some of the values may be extraneous roots and fail to satisfy the equation.

Example 1.

The equation $x + 1 = 3$ has only one root, $x = 2$. Multiply both sides by $x - 1$, obtaining $x^2 - 1 = 3x - 3$. The new equation has the roots $x = 2$ and $x = 1$, as can be verified by substitution. Thus, the extraneous root $x = 1$ has been introduced.

Example 2.

The equation $x + 1 = 3$ has only one root, $x = 2$. Square both sides, obtaining $x^2 + 2x + 1 = 9$. The new equation has the roots $x = 2$ and $x = -4$, the latter being an extraneous root introduced by squaring.

In solving an equation involving fractions, we usually clear of fractions by multiplying both sides of the equation by the L. C. D. If a common denominator other than the lowest is used as a multiplier, extraneous roots are likely to be introduced. If both sides of an equation are multiplied by the L. C. D. of the fractions involved, any root of the resulting equation is also a root of the original equation, provided it does not, when substituted, make any denominator of the original equation equal to zero.

Example 3.

Solve the equation

$$\frac{x}{x-3} = \frac{x-2}{x-4}.$$

Solution. Clear of fractions by multiplying both sides by the L. C. D., $(x-4)(x-3)$:

$$x^2 - 4x = x^2 - 5x + 6,$$
$$x = 6.$$

This satisfies the original equation, and is consequently a root.

Example 4.

Solve the equation

$$\frac{1}{x+1} + \frac{1}{x-1} = \frac{2}{x^2-1}.$$

Solution. Clear of fractions by multiplying both sides by the L. C. D., $(x+1)(x-1)$:

$$x - 1 + x + 1 = 2,$$
$$2x = 2, \qquad x = 1.$$

When we attempt to substitute the value $x = 1$, we obtain

$$\frac{1}{2} + \frac{1}{0} = \frac{2}{0}.$$

But (section 4) division by zero is excluded; hence 1 is not a root. In fact, the equation has no solution.

EXERCISES IV. D

Solve:

1. $\dfrac{2}{x-3} - \dfrac{3}{x+2} = 0.$

2. $\dfrac{2x}{x-3} - \dfrac{3}{x+2} = 2.$

3. $\dfrac{2x}{4x-3} - \dfrac{1}{3x+2} = \dfrac{1}{2}.$

4. $\dfrac{3x}{2x+7} + \dfrac{2x}{7x+5} = \dfrac{12}{13}.$

5. $\dfrac{2}{x^2-4} + \dfrac{1}{x+2} = \dfrac{3}{x-2}.$

6. $\dfrac{3}{x^2-25} = \dfrac{1}{x+5} + \dfrac{2}{x-5}.$

7. $\dfrac{20}{x^2-25} = \dfrac{1}{x+5} + \dfrac{2}{x-5}.$

8. $\dfrac{2x+3}{3x-2} = \dfrac{4x+1}{6x+1}.$

9. $\dfrac{2}{x+1} + \dfrac{7}{y-9} = 0,$

   $\dfrac{1}{x} + \dfrac{2}{y-1} = 0.$

10. $\dfrac{x}{x+2} + \dfrac{y}{y-2} = 2,$

    $\dfrac{3x+1}{2x} - \dfrac{2y+1}{y} = -\dfrac{1}{2}.$

11. Find the number which, when subtracted from the numerator and added to the denominator, changes the fraction $\frac{9}{19}$ into a fraction whose value is $\frac{1}{3}$.

12. If 4 is added to the numerator and subtracted from the denominator of a fraction, the value of the resulting fraction is $\frac{3}{4}$. If 4 is subtracted from the numerator and added to the denominator of the fraction, the value of the resulting fraction is $\frac{1}{4}$. What is the fraction?

13. A motorist set out from A at a certain rate and would have reached B in 6 hours if he had continued at this rate. However, when he had gone two-thirds of the way he had a flat tire which delayed him for an hour. During the remainder of the trip he drove at a speed which was 5 miles per hour faster than the rate at which he had started out, and arrived at B 48 minutes later than he had planned. How far is it from A to B and at what rate did he start out?

CHAPTER V

Exponents and Radicals

33. Fractional exponents.

According to the definition of a^m given in section 7, such expressions as $a^{1/2}$, a^0, and a^{-3} have no meaning, inasmuch as that discussion dealt only with the use of positive whole numbers as exponents. We find it desirable to give them meanings such that the laws of exponents, as developed in section 7, will still be valid.

By one of these laws, $(a^m)^2 = a^{2m}$. If we apply this same formula with $m = \frac{1}{2}$, we have $(a^{1/2})^2 = a$. Thus, if this law is to hold, we must have $a^{1/2} = \sqrt{a}$. In general, we define

$$a^{1/n} = \sqrt[n]{a}.$$

Example 1.

$$(27)^{1/3} = \sqrt[3]{27} = 3.$$

The symbol $\sqrt[n]{a}$ (nth root of a) is called a **radical.** The integer n is the **index** of the radical, the number a is the **radicand.**

As will be proved in Chapter XII, every number except zero has exactly n distinct nth roots, some or all of which may be imaginary numbers.

If n is odd, every real number a has just one real nth root, which is positive when a is positive, and negative when a is negative. For example, the real cube root of 8 is 2, the real cube root of -27 is -3.

If n is even, every positive number a has just two real

nth roots, one positive and one negative, with equal absolute values. For example, the two real fourth roots of 16 are ± 2.

If n is even and a is negative, the nth roots of a are all imaginary. For example, the square roots of -9 are imaginary numbers.

The **principal nth root** of a is the positive nth root of a when a is positive, and the negative nth root of a when a is negative and n is odd. When n is even and a is negative we do not define the principal nth root of a.

The symbol $a^{1/n} = \sqrt[n]{a}$ *will be restricted to mean the principal nth root of a when a is a real number different from zero.* As has been stated above, when a is negative and n is even the nth roots of a are all imaginary numbers. Such numbers will be discussed in section 44 of the present chapter and more fully in Chapter XII. For the present, in order to avoid them and to avoid ambiguous signs, we make the following stipulation: If the index of a radical is even, all literal numbers (except exponents) occurring under the radical sign represent positive numbers and are such that the radicand is positive.

We have then,

$$a^{m/n} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m.$$

Example 2.

$$(64)^{2/3} = (\sqrt[3]{64})^2 = 4^2 = 16.$$

34. Zero exponent.

If, in the relation $a^m/a^n = a^{m-n}$, we set $m = n$, we are led to the result

$$\frac{a^m}{a^m} = a^{m-m} = a^0.$$

But

$$\frac{a^m}{a^m} = 1, \qquad a \neq 0.$$

Consequently, we define

$$a^0 = 1, \qquad a \neq 0.$$

We assign no meaning whatever to the symbol 0^0.

Examples.

$$3^0 = 1, \qquad (3x)^0 = 1, \qquad 3x^0 = 3 \cdot 1 = 3.$$

35. Negative exponents.

Similarly, if the law $a^m/a^n = a^{m-n}$ is to hold for $m = 3$ and $n = 5$, for example, we must have

$$\frac{a^3}{a^5} = a^{3-5} = a^{-2}.$$

But

$$\frac{a^3}{a^5} = \frac{1}{a^2},$$

and if these two results are to be consistent, we must have $a^{-2} = \dfrac{1}{a^2}$. In general, we define

$$a^{-m} = \frac{1}{a^m}, \qquad a \neq 0.$$

Examples.

$$3^{-2} = \frac{1}{3^2} = \frac{1}{9}, \qquad 8^{-1/3} = \frac{1}{8^{1/3}} = \frac{1}{\sqrt[3]{8}} = \frac{1}{2}.$$

36. Summary of laws of exponents.

With the foregoing definitions of fractional, zero, and negative exponents, the following laws hold in general:

$$a^m a^n = a^{m+n}.$$

$$\frac{a^m}{a^n} = a^{m-n}, \qquad a \neq 0.$$

$$(a^m)^n = a^{mn}.$$

$$\sqrt[n]{a^m} = (\sqrt[n]{a})^m = a^{m/n}.$$

$$a^n b^n = (ab)^n.$$

EXERCISES V. A

Perform the indicated operations and simplify the results:

1. $(36)^{1/2}$.
2. $(36x^{36})^{1/2}$.
3. $(27a^{27})^{1/3}$.
4. $(4a^4b^8)^{1/2}$.
5. $(64a^{12})^{1/2}$.
6. $(64a^{12})^{1/3}$.
7. $(64a^{12})^{1/6}$.
8. $(16a^{16})^{1/4}$.
9. $5x^{-1}$.
10. $(5x)^{-1}$.
11. $\frac{1}{3}x^{1/3} \cdot \frac{1}{4}x^{1/4}$.
12. $\frac{1}{3}x^{1/3} \div \frac{1}{4}x^{1/4}$.
13. $(\frac{1}{27}\, x^{1/2})^{1/3}$.
14. $(\frac{1}{16}\, x^{1/2})^2$.
15. $\left(\frac{a}{b}\right)^{-2}$.
16. $\left(-\frac{a}{b}\right)^{-2}$.
17. $(\frac{1}{16}\, x^{1/2})^{-2}$.
18. $(\frac{1}{16}\, x^{-2})^{1/2}$.
19. $(\frac{1}{2}x^0)^3$.
20. $(\frac{1}{2}x^3)^0$.
21. $(64a^{12})^{2/3}$.
22. $(64a^{-3})^{-2/3}$.
23. $\left(\frac{27}{a^{27}}\right)^{2/3}$.
24. $\left(\frac{27}{a^{27}}\right)^{-2/3}$.
25. $\left(\frac{8a^{12}b^9c^{27}}{27x^3y^6}\right)^{2/3}$.
26. $\left(\frac{125x^6y^9z^{12}}{64a^0b^{15}}\right)^{-2/3}$.
27. $\left(\frac{125x^6y^9z^{12}}{64a^{-1}b^{-2}c^0}\right)^0$.
28. $\left(\frac{8^{-1}a^{-3}b^{-9}}{27c^{27}}\right)^{-1/3}$.
29. $\dfrac{3^{-3} - 4^{-4}}{2^{-2}}$.
30. $\dfrac{9^{1/2} - (16)^{-1/2}}{3^0 + 8^{1/3}}$.
31. $\dfrac{\frac{2}{3} \cdot 27 - (27)^{2/3}}{\frac{3}{4} \cdot 16 - (16)^{3/4}}$.
32. $\dfrac{(27)^{-2/3} + (16)^{-3/4}}{\frac{2}{3} \cdot 27 + \frac{3}{4} \cdot 16}$.
33. $\dfrac{x^{-1} - y^{-1}}{(x - y)^{-1}}$.
34. $\dfrac{(x + y)^{-2}}{x^{-2} + y^{-2}}$.
35. $\dfrac{a^0 + a^3}{a^0 - a^{-1} + a^{-2}}$.
36. $\dfrac{6a^{-2} + (ab)^{-1} - b^{-2}}{a^2b^{-2} + ab^{-1} - 2a^0}$.
37. $\dfrac{\left(\frac{a}{b} - c\right)^{-2} + \left(\frac{a}{b} + c\right)^{-2}}{\left(\frac{a}{b} + c\right)^{-2} - \left(\frac{a}{b} - c\right)^{-2}}$.
38. $\dfrac{\left(\frac{x}{y}\right)^{1/3} - \left(\frac{y}{x}\right)^{-1/3}}{\left(\frac{x}{y} - \frac{y}{x}\right)^{-1/3}}$.

39. $(2x^{3/2} + 5x^{1/2} - 3x^{-1/2})(3x^{3/2} - 4x^{1/2})$.
40. $(3x^{7/3}y^{-1/3} - 4x^{4/3}y^{2/3} + 7x^{1/3}y^{5/3})(5x^{2/3}y^{1/3} - 2x^{-1/3}y^{4/3})$.
41. $(x^{5/3} - 2x^{2/3} - 3x^{-1/3})(4x^{3/2} - 5x^{1/2})$.
42. $(5x^3 - 21x^2 + 20x - 6) \div (x^{1/2} - 3x^{-1/2})$.
43. $(6x^{5/2} + 7x^{3/2} - 29x^{1/2} + 12x^{-1/2}) \div (3x^{1/2} - 4x^{-1/2})$.

Any number may be written as the product of a number having a single nonzero digit to the left of the decimal point, and a positive or negative power of 10. A number so written is said to be written in **scientific notation**. Write each of the following numbers in scientific notation:

44. (a) 14,700,000; (b) 0.000,032,8.

SOLUTION. (a) $14{,}700{,}000 = 1.47 \times 10^7$;
(b) $0.000{,}032{,}8 = 3.28 \times 10^{-5}$.

45. 4,560,000.
46. 375,000,000.
47. 0.000,168.
48. 0.000,002,5.
49. $3\frac{1}{2}$ million.
50. $\frac{1}{8000}$.
51. 92,900,000 (number of miles from earth to sun).
52. 2,655,000 (number of foot-pounds in 1 kilowatt-hour).
53. 0.000,064,38 (wave length, in centimeters, of red light).
54. 0.000,000,373 (number of horsepower in 1 joule per hour).

Perform the indicated arithmetical operations and write the result in scientific notation. (It is best to write each number in scientific notation first.)

55. $26{,}000{,}000 \times 1{,}500{,}000$.
56. $0.000{,}03 \times 0.000{,}000{,}6$.
57. $17{,}000 \times 0.000{,}000{,}03$.
58. $15{,}000{,}000 \div 62{,}500$.
59. $16{,}000{,}000 \div 0.000{,}64$.
60. $0.000{,}009{,}6 \div 0.000{,}012$.
61. The speed of light is 186,000 miles per second. A light-year is the distance that light travels in one year. Find to three significant figures (see section 129) the number of miles in a light-year, expressing the result in scientific notation.

37. Removing factors from and introducing them into radicals.

A factor which is a perfect power may be removed from underneath a radical sign as in the following illustrations, in which the literal quantities are to be regarded as positive.

$$\sqrt{9x^3} = \sqrt{9x^2}\sqrt{x} = 3x\sqrt{x},$$
$$\sqrt{50x^3y^2} = \sqrt{25x^2y^2}\sqrt{2x} = 5xy\sqrt{2x},$$
$$\sqrt[3]{16x^6y^5z^4} = \sqrt[3]{8x^6y^3z^3}\sqrt[3]{2y^2z} = 2x^2yz\sqrt[3]{2y^2z}.$$

Inversely, a factor may be introduced under a radical sign, thus:

$$3x\sqrt{yz} = \sqrt{9x^2}\sqrt{yz} = \sqrt{9x^2yz}.$$

38. Reducing the index of a radical.

The following example illustrates how the index of a radical can sometimes be reduced:

$$\sqrt[4]{36x^2} = \sqrt[4]{(6x)^2} = (6x)^{2/4} = (6x)^{1/2} = \sqrt{6x}.$$

This shows one advantage of fractional exponents.

EXERCISES V. B

Remove all perfect powers from beneath radical signs:

1. $\sqrt{49a}$.
2. $\sqrt{75a^2b^3}$.
3. $\sqrt[3]{54ab^2c^3d^5e^7}$.
4. $\sqrt[3]{-16x^{16}}$.
5. $\sqrt[4]{64u^5v^7w^{14}}$.
6. $\sqrt{49x^{49}}$.
7. $\sqrt{0.50x^{25}}$.
8. $\sqrt[3]{0.250m^{24}n^{25}}$.
9. $\sqrt{12x^{-12}}$.
10. $\sqrt[3]{-16x^{-12}}$.
11. $\sqrt{\dfrac{27ab^2c^3}{4x^4y^6}}$.
12. $\sqrt{\dfrac{4a^2b^3c^5}{27x^3y^6}}$.
13. $\sqrt[5]{128x^7}$.
14. $\sqrt{10368x^{41}}$.

Introduce all coefficients under the radical signs:

15. $3\sqrt{xy}$.
16. $4xy^2\sqrt{xy}$.

17. $3ab^2\sqrt[3]{ab}$.

18. $2ab^2c^3\sqrt[4]{abc}$.

19. $\dfrac{1}{2x}\sqrt{14x}$.

20. $\dfrac{3x}{5y}\sqrt[3]{100xy}$.

21. $(x+y)\sqrt{\dfrac{x-y}{x+y}}$.

22. $(x-y)\sqrt{\dfrac{x^2+xy+y^2}{x-y}}$.

23. $a\sqrt[n]{a}$.

24. $a^2b^3\sqrt[n]{ab}$.

25. $x^m\sqrt[n]{x}$.

26. $x^m\sqrt[n]{x^p}$.

Reduce the indexes of the following radicals:

27. $\sqrt[4]{36a^2}$.

28. $\sqrt[6]{8a^3}$.

29. $\sqrt[6]{64a^3}$.

30. $\sqrt[6]{64a^2}$.

31. $\sqrt[4]{\dfrac{25a^2}{169b^2}}$.

32. $\sqrt[6]{\dfrac{27a^3}{343b^3}}$.

33. $\sqrt[12]{x^2y^4z^6}$.

34. $\sqrt[12]{x^4y^4z^8}$.

35. $\sqrt[12]{x^6y^6z^0}$.

36. $\sqrt[7n]{x^{2n}y^{3n}}$.

39. Rationalizing the denominator.

When a radical occurs in the denominator, the denominator can be **rationalized** (that is, freed from radicals) as shown in the examples below:

Example 1.

$$\frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3}, \quad \text{or} \quad \frac{1}{3}\sqrt{3}.$$

The principal advantage of rationalizing the denominator is in finding the numerical value of the given expression. Thus, $\sqrt{3} = 1.7321$, and in example 1 it is easier to evaluate $\dfrac{1.7321}{3}$ than $\dfrac{1}{1.7321}$. There is no advantage, however, if one is using a calculating machine or logarithms.

Example 2.

$$\frac{1}{\sqrt[3]{2}} = \frac{1}{\sqrt[3]{2}}\cdot\frac{\sqrt[3]{2^2}}{\sqrt[3]{2^2}} = \frac{\sqrt[3]{4}}{2}.$$

We rationalize in the same way when a fraction occurs under a radical sign.

Example 3.

$$\sqrt{\frac{2}{3}} = \sqrt{\frac{2}{3}\cdot\frac{3}{3}} = \frac{\sqrt{6}}{3}, \quad \text{or} \quad \frac{1}{3}\sqrt{6}.$$

When the denominator is $a\sqrt{b} + c\sqrt{d}$, we multiply numerator and denominator of the fraction by $a\sqrt{b} - c\sqrt{d}$; if the denominator is $a\sqrt{b} - c\sqrt{d}$ we multiply by $a\sqrt{b} + c\sqrt{d}$. This will eliminate the radicals in the denominator, since the product of the sum and difference of two numbers is equal to the difference of their squares.

Example 4.

$$\frac{1+\sqrt{2}}{\sqrt{3}+\sqrt{2}} = \frac{1+\sqrt{2}}{\sqrt{3}+\sqrt{2}}\cdot\frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}-\sqrt{2}} = \frac{\sqrt{3}-\sqrt{2}+\sqrt{2}\sqrt{3}-\sqrt{2}\sqrt{2}}{(\sqrt{3})^2-(\sqrt{2})^2}$$

$$= \frac{\sqrt{3}-\sqrt{2}+\sqrt{6}-2}{3-2} = \sqrt{3}-\sqrt{2}+\sqrt{6}-2.$$

EXERCISES V. C

Rationalize the denominators of the following fractions or remove the denominators from the radical signs:

1. $\dfrac{2}{\sqrt{3}}$. 2. $\dfrac{x}{\sqrt{y}}$. 3. $\dfrac{3}{\sqrt[3]{2}}$.

4. $\dfrac{3}{\sqrt[3]{4}}$. 5. $\dfrac{x}{\sqrt[3]{y}}$. 6. $\dfrac{x}{\sqrt[3]{y^2}}$.

7. $\dfrac{2}{\sqrt{a+b}}$. 8. $\sqrt{\dfrac{3x}{5y}}$. 9. $\sqrt[3]{\dfrac{3x}{2y^2}}$.

10. $\sqrt[3]{\dfrac{3x^2}{4y}}$. 11. $\dfrac{1}{\sqrt[5]{a^2}}$. 12. $\dfrac{1}{\sqrt[5]{4a^3}}$.

13. $\dfrac{1}{\sqrt{7}-\sqrt{2}}$. 14. $\dfrac{4}{\sqrt{5}+\sqrt{3}}$. 15. $\dfrac{1}{\sqrt{x}+\sqrt{y}}$.

16. $\dfrac{3+\sqrt{2}}{3-\sqrt{2}}$. 17. $\dfrac{\sqrt{11}-\sqrt{3}}{\sqrt{11}+\sqrt{3}}$. 18. $\dfrac{a+\sqrt{b}}{a-\sqrt{b}}$.

19. $\dfrac{\sqrt{a}-\sqrt{b}}{\sqrt{a}+\sqrt{b}}$. 20. $\dfrac{\sqrt{7}-\sqrt{3}}{\sqrt{5}+\sqrt{2}}$. 21. $\dfrac{1}{1-\sqrt{2}+\sqrt{3}}$.

22. $\dfrac{1}{1+\sqrt{2}+\sqrt{3}}$. 23. $\dfrac{1}{\sqrt{2}+\sqrt{3}-\sqrt{5}}$.

24. $\dfrac{1}{\sqrt{2}-\sqrt{3}-\sqrt{5}}$.

25. Show that the denominator $1 + x^{1/3}$ (i.e., $1+\sqrt[3]{x}$) can be rationalized by the multiplier $1 - x^{1/3} + x^{2/3}$.

Rationalize the denominator of

26. $\dfrac{1}{a^{1/3}+b^{1/3}}$. 27. $\dfrac{1}{a^{1/3}-b^{1/3}}$.

Find the decimal values of the following expressions, correct to thousandths, first rationalizing the denominators:

28. $\dfrac{1}{\sqrt{3}}$. 29. $\dfrac{3}{\sqrt{2}}$. 30. $\sqrt{\dfrac{2}{3}}$.

31. $\dfrac{2+\sqrt{5}}{2-\sqrt{5}}$. 32. $\dfrac{10}{\sqrt{7}+\sqrt{2}}$. 33. $\dfrac{\sqrt{7}+\sqrt{5}}{\sqrt{5}-\sqrt{2}}$.

40. Reduction of radical expressions to simplest form.

An expression containing radicals is usually regarded as being in its simplest form when

(i) *no factor can be removed from a radical,*
(ii) *no index can be reduced,*
(iii) *there are no fractions under a radical sign,*
(iv) *there are no radicals in a denominator.*

EXERCISES V. D

Reduce to simplest form:

1. $\sqrt{18a^3b^4c^5}$. 2. $\sqrt{75x^5y^{15}}$. 3. $\sqrt[3]{16x^3y^5z^{16}}$.

4. $\sqrt[3]{250a^4b^5c^7}$. 5. $\sqrt{\dfrac{8a^3}{5b}}$. 6. $\sqrt{\dfrac{45x^5}{7y}}$.

7. $\frac{2x}{\sqrt{8xy}}$.
8. $\frac{5u}{\sqrt{162uv}}$.
9. $\sqrt[3]{\frac{135a}{7b}}$.
10. $\sqrt[3]{\frac{11x}{36y^2}}$.
11. $\frac{1}{\sqrt[3]{4xy^2}}$.
12. $\frac{1}{\sqrt[5]{-8a^2b^3}}$.
13. $\frac{1}{a^{1/2}+b^{1/2}}$.
14. $\frac{2}{\sqrt[4]{36a^2}}$.
15. $\frac{1}{\sqrt{11}-\sqrt{5}}$.
16. $\frac{\sqrt{7}-\sqrt{6}}{3\sqrt{7}+2\sqrt{6}}$.
17. $(49x)^{-1/3}$.
18. $(8x^7)^{-2/3}$.
19. $\frac{7}{\sqrt{3}-\sqrt{\frac{2}{3}}}$.
20. $\frac{\sqrt{5}-\frac{1}{\sqrt{3}}}{\sqrt{5}+\sqrt{3}}$.
21. $\frac{\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{3}}}{\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}}$.
22. $\frac{\frac{3}{\sqrt{2}}+\frac{2}{\sqrt{3}}}{\frac{5}{\sqrt{2}}-\frac{4}{\sqrt{3}}}$.
23. $\frac{\sqrt{\frac{1}{2}}+\sqrt{\frac{1}{3}}}{\sqrt{\frac{1}{2}+\frac{1}{3}}}$.
24. $\frac{\sqrt{\frac{1}{2}-\frac{1}{3}}}{\sqrt{\frac{1}{2}}-\sqrt{\frac{1}{3}}}$.
25. $\sqrt[n]{x^{n+3}y^{n+5}}$.
26. $\sqrt[n]{x^{3n}y^{5n}}$.
27. $\sqrt[2n]{a^nb^{3n}}$.
28. $\sqrt[n]{a^{2n+3}b^{3n+2}}$.
29. $\frac{1+\sqrt{3}}{1+\sqrt{2}+\sqrt{3}}$.
30. $\frac{1-\sqrt{5}}{1+\sqrt{3}-\sqrt{5}}$.

41. Addition and subtraction of radicals.

Radicals can be added or subtracted only when they have the same index and the same number under the radical sign.

Example.

$$2\sqrt{3}+5\sqrt{3}+3\sqrt{2}-\sqrt{3}=(2+5-1)\sqrt{3}+3\sqrt{2}$$
$$=6\sqrt{3}+3\sqrt{2}.$$

42. Multiplication of radicals.

Radicals can be multiplied only if they have the same index. (But see example 2 following.)

Example 1.

$$\sqrt{3}\sqrt[3]{5}=3^{1/2}\cdot 5^{1/3}=3^{3/6}\cdot 5^{2/6}=\sqrt[6]{3^3}\sqrt[6]{5^2}=\sqrt[6]{3^3\cdot 5^2}=\sqrt[6]{675}.$$

Example 2.

$$\sqrt{3}\sqrt[4]{3} = 3^{1/2} \cdot 3^{1/4} = 3^{1/2+1/4} = 3^{3/4} = \sqrt[4]{3^3} = \sqrt[4]{27}.$$

43. Division of radicals.

Division of radicals is usually best effected by the process of rationalizing the denominator. Sometimes, however, this is not necessary. (See examples 1 and 3 below.)

Example 1.

$$\frac{\sqrt{6}}{\sqrt{2}} = \sqrt{\frac{6}{2}} = \sqrt{3}.$$

Example 2.

$$\frac{\sqrt{6}}{\sqrt{5}} = \frac{\sqrt{6}}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{\sqrt{30}}{5}.$$

Example 3.

$$\frac{\sqrt{6}}{\sqrt[3]{6}} = \frac{6^{1/2}}{6^{1/3}} = 6^{1/2-1/3} = 6^{1/6} = \sqrt[6]{6}.$$

Example 4.

$$\frac{\sqrt{6}}{\sqrt[3]{2}} = \frac{6^{1/2}}{2^{1/3}} = \frac{6^{3/6}}{2^{2/6}} = \frac{\sqrt[6]{6^3}}{\sqrt[6]{2^2}} = \sqrt[6]{\frac{6^3}{2^2}} = \sqrt[6]{54}.$$

Example 5.

$$\frac{\sqrt{6}}{\sqrt[3]{5}} = \frac{6^{1/2}}{5^{1/3}} = \frac{6^{3/6}}{5^{2/6}} = \sqrt[6]{\frac{6^3}{5^2}} = \sqrt[6]{\frac{6^3 \cdot 5^4}{5^2 \cdot 5^4}}.$$

$$= \frac{1}{5}\sqrt[6]{6^3 \cdot 5^4} = \frac{1}{5}\sqrt[6]{135{,}000}.$$

EXERCISES V. E

Simplify, and combine like terms:

1. $8\sqrt{2} - 3\sqrt{2} + 4\sqrt{2}$.
2. $x\sqrt{y} + z\sqrt{y} - 3\sqrt{y}$.
3. $8\sqrt{2} - 2\sqrt{8}$.
4. $3\sqrt[4]{36} - 2\sqrt[6]{216} + \sqrt{216}$.

5. $\sqrt{\dfrac{2x}{3}} + \sqrt{\dfrac{3x}{2}}$.

6. $\left(\dfrac{1}{3}\right)^{1/3}$.

7. $\sqrt{64} + \sqrt[3]{64} + \sqrt[4]{64} + \sqrt[5]{64} + \sqrt[6]{64}$.

8. $\sqrt[3]{\dfrac{a^2}{b}} + \sqrt{\dfrac{b^2}{a}} + \sqrt[3]{\dfrac{a}{b^2}} + \sqrt[3]{\dfrac{b}{a^2}}$.

9. $2^{-1/2} - \left(\dfrac{1}{2}\right)^{-2} + 4^{-1/4} - \left(-\dfrac{1}{4}\right)^{-4}$.

10. $\left(\dfrac{2}{5}\right)^{-2/5} + \dfrac{1}{(0.4)^{0.4}} + \sqrt[5]{6400} + \sqrt[5]{0.002}$.

11. $\sqrt{\dfrac{3}{5}} + \sqrt[4]{225} - \dfrac{3}{5}\sqrt[4]{\dfrac{25}{9}} - \dfrac{\dfrac{5}{3}}{\sqrt{15}}$.

12. $2a^{-1/3} - 3a^{4/3} + 4\left(\dfrac{1}{a}\right)^{1/3} - \dfrac{1}{a^{2/3}} - 5\left(\dfrac{1}{a^2}\right)^{1/3} + 6a^{-2/3}$.

13. $\sqrt{22\frac{1}{2} \cdot x^3} + 22\frac{1}{2} \cdot x^2\sqrt{10x^{-1}} + \sqrt{10^{-1}x} + \sqrt{1000x^{-3}}$.

14. $(x^{1/3})^{-1/2} + x^{1/3} \cdot x^{1/2} + x^{1/3} \div x^{1/2}$.

Perform the indicated operations and simplify results

15. $\sqrt{6} \cdot \sqrt{13}$.

16. $\sqrt{6} \cdot \sqrt{15}$.

17. $\sqrt[3]{7x^2}\sqrt[4]{49x^6}$.

18. $\sqrt{3} \cdot \sqrt[3]{3}$.

19. $\sqrt{2} \cdot \sqrt[3]{3}$.

20. $\sqrt[4]{2} \cdot \sqrt[3]{5}$.

21. $\sqrt{2} \cdot \sqrt[3]{3} \cdot \sqrt[4]{4} \cdot \sqrt[6]{6}$.

22. $\sqrt[4]{8} \cdot \sqrt[6]{16} \cdot \sqrt[16]{32}$.

23. $3^{-1/3} \cdot 9^{-1/9} \cdot (27)^{-1/27}$.

24. $5^{-1/5} \cdot (25)^{-2/5} \cdot (125)^{-3/5}$.

25. $\dfrac{1}{3}\sqrt[3]{a} \cdot \dfrac{1}{4}\sqrt[4]{a}$.

26. $\left(\dfrac{2}{3}\right)^{2/3} \cdot \left(\dfrac{3}{4}\right)^{3/4}$.

27. $(3\sqrt{3})^3$.

28. $(2\sqrt[3]{2})^3$.

29. $(\sqrt{3} + \sqrt[3]{2})(\sqrt{3} - \sqrt[3]{2})$.

30. $(\sqrt{3} - \sqrt[3]{2})^2$.

31. $(\sqrt{6} - 2\sqrt[3]{3})(\sqrt{2} + 3\sqrt[3]{2})$.

32. $(\sqrt{5} - \sqrt[3]{6})(\sqrt[3]{5} + \sqrt{6})$.

33. $(\sqrt{x} + \sqrt[3]{x} + \sqrt[4]{x})(\sqrt{x} - \sqrt[3]{x} + \sqrt[4]{x})$.

34. $\dfrac{-b + \sqrt{b^2 - 4ac}}{2a} \cdot \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$.

35. $\left(x - \dfrac{5 + 2\sqrt{3}}{2}\right)\left(x - \dfrac{5 - 2\sqrt{3}}{2}\right)$.

36. $\sqrt{115} \div \sqrt{23}$.

37. $\sqrt[3]{125} \div \sqrt[3]{5}$.

38. $\sqrt{23} \div \sqrt[3]{23}$.
39. $\sqrt[3]{5} \div \sqrt{5}$.
40. $\sqrt{5} \div \sqrt[3]{25}$.
41. $\sqrt{5} \div \sqrt{6}$.
42. $\sqrt{22} \div \sqrt{6}$.
43. $\sqrt[3]{4ab^2c^2} \div \sqrt[4]{2a^3b^2c}$.
44. $\sqrt{\frac{x}{y}} \div \sqrt{\frac{z}{x}}$.
45. $\sqrt[3]{\frac{x}{y}} \div \sqrt{\frac{z}{x}}$.
46. $\sqrt[4]{\frac{a^3}{b}} \div \sqrt[3]{\frac{a}{b^2}}$.
47. $\sqrt[3]{\frac{a^2}{b}} \div \sqrt[4]{\frac{a^2}{b^3}}$.
48. $(\sqrt[3]{3x} + \sqrt[4]{4x} + \sqrt[6]{6x}) \div \sqrt{2x}$.
49. Find the value of $x^2 - 6x + 7$ if $x = 3 - \sqrt{2}$.
50. Find the value of $4x^2 + 6x - 3$ if $x = \frac{3 - \sqrt{5}}{4}$.
51. Find the value of $9x^2 - 24x + 11$ if $x = \frac{4 - \sqrt{5}}{3}$.
52. Find the value of $x^2 + px + q$ if $x = -\frac{1}{2}(p + \sqrt{p^2 - 4q})$.
53. Find the value of $x^2 + px + q$ if $x = \frac{1}{2}(p + \sqrt{p^2 - 4q})$.
54. Find the value of $ax^2 + bx + c$ if $x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$.
55. Find the value of $x^3 + 12x - 12$ if $x = 2\sqrt[3]{2} - \sqrt[3]{4}$.

44. Complex numbers.*

A negative number has no real square root, for no positive or negative number, when squared, will yield a negative number. Consequently, in order to solve certain equations, for example $x^2 + 4 = 0$, it is necessary to introduce a new kind of number, called an **imaginary number.**

The imaginary unit is the number i having the property

$$i^2 = -1.$$

It is assumed that this new number obeys all the laws of addition and multiplication assumed for real numbers.

From the definition of i, it is seen that we may write

$$\sqrt{-1} = i, \sqrt{-4} = \sqrt{4}\sqrt{-1} = 2i,$$
$$\sqrt{-3} = \sqrt{3}\sqrt{-1} = i\sqrt{3}.$$

* See also Chapter XII.

A number of the form $a + bi$, in which a and b are real numbers, is called a **complex number,** for example, $2 - 3i$. The number a is called the **real part,** and bi is called the **imaginary part** of the complex number, b being the coefficient of the imaginary part. In the example given, 2 is the real part and $-3i$ the imaginary part. If $b \neq 0$, the complex number is called an **imaginary number.** Thus, $2 - 3i$ and $5i$ are imaginary numbers. If $b \neq 0$ and $a = 0$, the complex number reduces to the form bi, which is called a **pure imaginary number,** for example, $5i$. If $b = 0$, the complex number reduces to the real number a, for example, 2.

Thus, complex numbers include real numbers and imaginary numbers as special cases. For example, $2 - 3i$, 2 (that is, $2 + 0i$), and $5i$ (that is, $0 + 5i$) are all complex numbers. As already stated, the first and third of these are imaginary, the third is pure imaginary; the second number, namely 2, is real.

It should be noted that $a + bi$ and $a - bi$ are called **conjugate complex numbers,** either being the conjugate of the other.

By definition, we add or subtract complex numbers by adding or subtracting their real parts and imaginary parts separately.

Example 1.

$$(2 + 5i) + (6 - 4i) = (2 + 6) + (5 - 4)i = 8 + i.$$
$$(2 + 5i) - (6 - 4i) = (2 - 6) + (5 + 4)i = -4 + 9i.$$

By definition, we multiply complex numbers as in the following example:

Example 2.

$$\begin{aligned}(2 + i)(1 - 3i) &= 2 - 5i - 3i^2 \\ &= 2 - 5i + 3 \text{ (since } i^2 = -1) \\ &= 5 - 5i.\end{aligned}$$

NOTE. When operating with an imaginary number always use i rather than $\sqrt{-1}$, otherwise incorrect results may be obtained, especially in multiplication.

To divide one complex number by another, we write the quotient as a fraction and multiply numerator and denominator by the conjugate of the denominator

Example 3.

$$(2 + i) \div (1 - 3i) = \frac{2 + i}{1 - 3i} \cdot \frac{1 + 3i}{1 + 3i}$$

$$= \frac{2 + 7i + 3i^2}{1 - 9i^2} = \frac{2 + 7i - 3}{1 + 9}$$

$$= \frac{-1 + 7i}{10}, \quad \text{or} \quad -\frac{1}{10} + \frac{7}{10}i.$$

EXERCISES V. F

Reduce to the form $a + bi$:

1. $6 + \sqrt{-9}$. **2.** $4 + \sqrt{-5}$. **3.** $7 - \sqrt{-6}$.
4. $2 + \sqrt{-\frac{1}{2}}$. **5.** $\frac{9}{25} + \sqrt{-\frac{9}{25}}$. **6.** $\frac{2}{7} - \sqrt{-\frac{2}{7}}$.
7. $-\frac{3}{4} + \sqrt{-\frac{1}{4}}$. **8.** $-\frac{1}{4} + \sqrt{-\frac{3}{4}}$. **9.** $\frac{4}{3} + \sqrt{-\frac{4}{3}}$.
10. $\frac{1}{22} - \sqrt{-\frac{3}{11}}$. **11.** $\sqrt[3]{-8} + \sqrt{-8}$. **12.** $\sqrt[3]{-9} - \sqrt{-9}$.

Perform the indicated operations and simplify:

13. $(4 + 3i) + (2 + 5i)$. **14.** $(6 - 3i) + (7 + 8i)$.
15. $(-2 - 9i) + (3 + 3i)$. **16.** $(5 + 4i) + (-5 + i)$.
17. $(7 - 9i) + (8 - 6i)$. **18.** $(6 + \sqrt{-4}) + (10 + \sqrt{-9})$.
19. $(5 + \sqrt{-2}) + (3 + 4\sqrt{-2})$.
20. $(7 - 2\sqrt{-3}) + (-1 - 5\sqrt{-3})$.
21. $(\frac{3}{2} + \sqrt{-\frac{2}{3}}) + (\frac{2}{3} + \sqrt{-\frac{3}{2}})$.
22. $(\frac{1}{2} - \sqrt{-2}) + (2 - \sqrt{-\frac{1}{2}})$.
23. $(\sqrt{3} + 5\sqrt{-2}) + (\sqrt{5} - 3\sqrt{-2})$.
24. $(3\sqrt{5} + 2\sqrt{-5}) + (2\sqrt{5} - 5\sqrt{-3})$.
25. $(8 + 7i) - (3 + 4i)$. **26.** $(6 - 2i) - (9 + 5i)$.
27. $(-11 + \sqrt{-3}) - (2 - 5\sqrt{-3})$.

28. $(\sqrt{2} - \sqrt{-2}) - (\sqrt{8} - \sqrt{-8})$.
29. $(2\sqrt{3} + 3\sqrt{-2}) - (-3\sqrt{2} + 2\sqrt{-3})$.
30. $(\sqrt[3]{-27} - \sqrt{-27}) - (\sqrt[3]{-3} - \sqrt{-3})$.
31. $(5 + 3i) - (6 - 4i) + (-8 + 7i)$.
32. $(-4 + 3\sqrt{-3}) - (9 - 5\sqrt{-3}) - (12 - \sqrt{-12})$.
33. $(\frac{2}{5} + \sqrt{-\frac{2}{5}}) + (\frac{5}{2} - \sqrt{-\frac{5}{2}}) - (\frac{1}{10} + \sqrt{-\frac{1}{10}})$.
34. $(2 - \sqrt{-2}) - (\sqrt{3} - \sqrt{-3}) - (\sqrt{4} - \sqrt{-4})$.
35. $(3 + 2i)(4 + 5i)$. 36. $(8 - 3i)(10 + 7i)$.
37. $(2 + i\sqrt{5})(7 + 6\sqrt{5} \cdot i)$.
38. $(5 + 3i)(5 - 3i)$.
39. $(3 - 2\sqrt{-3})(6 - 5\sqrt{-3})$.
40. $(2\sqrt{3} + 3\sqrt{-2})(3\sqrt{2} - 2\sqrt{-3})$.
41. $(12 + 5\sqrt{2} \cdot i)(12 - 5\sqrt{2} \cdot i)$.
42. $(5\sqrt{14} - 3\sqrt{6} \cdot i)(3\sqrt{35} - 2\sqrt{15} \cdot i)$.
43. $(\frac{1}{6} - i\sqrt{\frac{2}{3}})(\frac{5}{6} - i\sqrt{\frac{2}{3}})$. 44. $(\frac{5}{6} - i\sqrt{\frac{2}{3}})^2$.
45. $(a + bi)(a - bi)$. 46. $(a + bi)^2$.
47. $(1 + i)^3$. 48. $(3 + 2i)^3$.
49. $\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^3$. 50. $\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)^3$.
51. $(4 + 5i) \div (2 + 3i)$. 52. $(8 - 7i) \div (3 + 5i)$.
53. $(-2 + 3i) \div (7 - 6i)$. 54. $(10 - 7i) \div (3 - 8i)$.
55. $(6 - 7\sqrt{6} \cdot i) \div (5 + 2\sqrt{6} \cdot i)$.
56. $(3 + 2\sqrt{-5}) \div (8 - 3\sqrt{-5})$.
57. $(2\sqrt{6} + 3\sqrt{2} \cdot i) \div (3\sqrt{6} - 5\sqrt{2} \cdot i)$.
58. $(3\sqrt{7} - 2\sqrt{5} \cdot i) \div (4\sqrt{7} - 9\sqrt{5} \cdot i)$.
59. $\left(1 + \frac{\sqrt{3}}{2}i\right) \div \left(1 - \frac{\sqrt{3}}{2}i\right)$.
60. $(\sqrt{\frac{2}{3}} + i\sqrt{\frac{3}{2}}) \div (\sqrt{\frac{3}{2}} + i\sqrt{\frac{2}{3}})$.
61. $(\sqrt{2} + i\sqrt{3}) \div (1 + \sqrt{3} + i\sqrt{2})$.
62. $1 \div (a + bi)$. 63. $1 \div (a - bi)$.
64. $(a + bi) \div (a - bi)$. 65. $1 \div (a - bi)^2$.
66. Find the value of $4x^2 - 12x + 13$ if $x = \frac{3 - 2i}{2}$.
67. Find the value of $4x^2 + 20x + 172$ if $x = \frac{-5 + 7\sqrt{3} \cdot i}{2}$.

CHAPTER VI

Quadratic Equations

45. Quadratic equations.

A **quadratic equation** (in one unknown) is one in which the highest power of the unknown occurring is two. The first-degree term and the constant term may or may not be present. The following are quadratic equations:

$$3x^2 - 2x + 4 = 0, \quad x^2 + 5 = 0, \quad 2x^2 + 3x = 0, \quad 2x^2 = 0.$$

46. Solution by factoring.

Frequently a quadratic equation can be solved by factoring.

Example 1.

Solve $x^2 - 5x + 6 = 0$.

SOLUTION. In factored form, the equation is

$$(x - 2)(x - 3) = 0.$$

The product is zero if and only if one of the factors is zero. Therefore, all the roots can be found by equating each factor in turn to zero. Thus,

$$x - 2 = 0, \qquad x = 2.$$
$$x - 3 = 0, \qquad x = 3.$$

CHECK. $2^2 - 5 \cdot 2 + 6 = 4 - 10 + 6 = 0.$
$3^2 - 5 \cdot 3 + 6 = 9 - 15 + 6 = 0.$

Example 2.

Solve $3x^2 + 2x = 0.$

SOLUTION. $x(3x + 2) = 0.$

$$x = 0.$$

$$3x + 2 = 0, \qquad x = -\frac{2}{3}.$$

The roots are 0 and $-\frac{2}{3}$, as it can be readily verified that these numbers satisfy the equation.

47. First-degree term missing.

If the first-degree term is missing the equation can easily be solved.

Example 1.

Solve $4x^2 - 3 = 0.$

SOLUTION. $4x^2 = 3.$

$$x^2 = \frac{3}{4}.$$

$$x = \pm\sqrt{\frac{3}{4}} = \pm\frac{\sqrt{3}}{2} \quad \left(\text{or } \pm\frac{1}{2}\sqrt{3}\right).$$

(The sign $\pm$ is read "plus or minus.")

Example 2.

Solve $4x^2 + 3 = 0.$

SOLUTION. $4x^2 = -3.$

$$x^2 = -\frac{3}{4}.$$

$$x = \pm\sqrt{\frac{-3}{4}} = \pm\frac{\sqrt{-3}}{2} = \pm\frac{i\sqrt{3}}{2}.$$

48. Completing the square.

A quadratic equation can always be solved by completing the square.

Example.

Solve
$$3x^2 - 2x - 4 = 0.$$

SOLUTION.

Transpose the constant term:

$$3x^2 - 2x = 4.$$

Divide both sides by the coefficient of x^2:

$$x^2 - \frac{2}{3}x = \frac{4}{3}.$$

To complete the square of the left side, *add the square of half the coefficient of* x, namely, $\left(-\frac{1}{3}\right)^2 = \frac{1}{9}$:

$$x^2 - \frac{2}{3}x + \frac{1}{9} = \frac{1}{9} + \frac{4}{3},$$

$$\left(x - \frac{1}{3}\right)^2 = \frac{13}{9}.$$

Take the square root of both sides:

$$x - \frac{1}{3} = \frac{\sqrt{13}}{3} \quad \text{or} \quad x - \frac{1}{3} = -\frac{\sqrt{13}}{3},$$

$$x = \frac{1}{3} + \frac{\sqrt{13}}{3} \quad \text{or} \quad x = \frac{1}{3} - \frac{\sqrt{13}}{3}.$$

We usually write

$$x - \frac{1}{3} = \pm\frac{\sqrt{13}}{3}, \qquad x = \frac{1}{3} \pm \frac{\sqrt{13}}{3} = \frac{1 \pm \sqrt{13}}{3}.$$

49. Solution by formula.

The quadratic equation

$$ax^2 + bx + c = 0 \qquad (a \neq 0) \tag{1}$$

is representative of all quadratic equations. For by giving the proper values to a, b, c we can reproduce any quadratic equation whatever. Thus, (1) becomes $3x^2 - 2x - 4 = 0$ if we set $a = 3, b = -2, c = -4$. We can, however, solve (1) by completing the square, and obtain a formula by which any quadratic equation may be solved by mere substitution.

Transpose c: $\qquad ax^2 + bx = -c.$

Divide by a: $\qquad x^2 + \frac{b}{a}x = -\frac{c}{a}.$

Add to both sides the square of half the coefficient of x, namely, $\left(\frac{1}{2}\frac{b}{a}\right)^2 = \left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2}$:

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}.$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}.$$

Take the square root of both sides:

$$x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}.$$

Subtract $\frac{b}{2a}$ from both sides:

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a},$$

or
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{2}$$

That the two numbers

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \qquad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \tag{3}$$

are roots of the quadratic equation (1) can be verified by actual substitution.

EXERCISE

Verify, by actual substitution, that x_1 and x_2, as given by (3), are roots of (1).

Example 1.

Solve the equation $3x^2 - 2x - 4 = 0$.

SOLUTION. $a = 3, \quad b = -2, \quad c = -4.$

Substitute in (2):

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \cdot 3 \cdot (-4)}}{2 \cdot 3}$$
$$= \frac{2 \pm \sqrt{4 + 48}}{6} = \frac{2 \pm \sqrt{52}}{6}$$
$$= \frac{2 \pm 2\sqrt{13}}{6} = \frac{1 \pm \sqrt{13}}{3}.$$

Example 2.

Solve the equation $kx^2 + 3kx - 2x + 4 = k$.

SOLUTION. Rearrange the equation in the form

$$kx^2 + (3k - 2)x + (4 - k) = 0.$$

Then $\quad a = k, \quad b = 3k - 2, \quad c = 4 - k.$

Substitute in formula (2):

$$x = \frac{-3k + 2 \pm \sqrt{(3k - 2)^2 - 4k(4 - k)}}{2k}$$

$$= \frac{-3k + 2 \pm \sqrt{13k^2 - 28k + 4}}{2k}.$$

EXERCISES VI. A

Solve:

1. $x^2 - 9x + 14 = 0.$
2. $x^2 - 5x - 14 = 0.$
3. $x^2 + 8x + 12 = 0.$
4. $x^2 + x - 12 = 0.$
5. $x^2 - 6x + 5 = 0.$
6. $x^2 + 4x + 4 = 0.$
7. $y^2 - 8y + 16 = 0.$
8. $z^2 + 6z - 16 = 0.$
9. $x^2 - 2x + 5 = 0.$
10. $x^2 - 5x + 4 = 0.$
11. $x^2 - 10x + 24 = 0.$
12. $x^2 - 11x + 24 = 0.$
13. $t^2 - 126t + 125 = 0.$
14. $t^2 - 125t - 126 = 0.$
15. $x^2 - 49 = 0.$
16. $x^2 - 49x = 0.$
17. $x^2 + 49 = 0.$
18. $x^2 - 5 = 0.$
19. $x^2 + 8 = 0.$
20. $x(x + 3) = 18.$
21. $x^2 - 28x + 96 = 0.$
22. $x^2 - 9x - 162 = 0.$
23. $x^2 - 12x + 22 = 0.$
24. $x^2 + 6x + 25 = 0$
25. $x^2 + 5x + 7 = 0.$
26. $x^2 + x + 1 = 0.$
27. $6x^2 + 7x - 3 = 0.$
28. $10x^2 - 29x - 21 = 0.$
29. $12x^2 + 17x + 6 = 0.$
30. $2x^2 + 10x + 9 = 0.$
31. $4x^2 - 11x + 11 = 0.$
32. $9x^2 + 24x + 25 = 0.$
33. $25x^2 - 20x + 1 = 0$
34. $4x^2 - 12x - 19 = 0.$
35. $5x^2 - 2x + 5 = 0.$
36. $45x^2 - 54x - 8 = 0.$
37. $21x^2 - 29x - 10 = 0.$
38. $3x^2 - 2x + 7 = 0.$
39. $5x^2 - 10x + 3 = 0.$
40. $56x^2 - 25x - 4 = 0.$
41. $17x^2 - 16x - 3 = 0.$
42. $15x^2 - 13x + 3 = 0.$
43. $11x^2 - x\sqrt{5} - 1 = 0.$
44. $13x^2 + 2\sqrt{78} \cdot x + 6 = 0.$
45. $5x^2 - 10\sqrt{7} \cdot x + 4 = 0.$
46. $9x^2 + 6\sqrt{2} \cdot x + 5 = 0.$
47. $375x^2 - 725x - 350 = 0.$
48. $54x^2 + 171x + 135 = 0.$
49. $\dfrac{x + 1}{x - 2} = \dfrac{3x - 1}{x + 2}.$
50. $\dfrac{4x - 3}{2x - 1} = \dfrac{5x + 3}{3x + 1}.$

51. $\dfrac{3x-8}{x-2} = \dfrac{5x-2}{x+5}$.

52. $\dfrac{x-1}{x-3} + \dfrac{x-3}{x-5} = 4$.

53. $x^2 - 0.9x + 0.2 = 0$.

54. $x^2 - 0.17x - 0.02 = 0$.

Solve for x:

55. $x^2 - 6x + 9 = k$.

56. $x^2 - 6x + 9 = kx$.

57. $x^2 - 6x + 9 = kx^2$.

58. $x^2 - 5x + 9 = kx^2 + k^2x$.

59. $x^2 - 8xy + 12y^2 = 0$.

60. $12x^2 - 9xy - 3y^2 = 0$.

61. $x^2 - 8xy + 14y^2 = 0$.

62. $x^2 - 14xy + 74y^2 = 0$.

63. $x^2 + 4xy + 10y^2 = 0$.

64. $9x^2 - 12xy - 41y^2 = 0$.

65. $16x^2 - 34xy - 15y^2 = 0$.

66. $16x^2 + 24xy + 9y^2 = 0$.

67. $x^2 + y^2 - 2x + 4y = 0$.

68. $2x^2 - 3y^2 + 6x + 12y = 0$.

69. $x^2 + 4xy + 6y^2 - 8x - 12y + 6 = 0$.

70. Solve equations 59–69 for y.

71. The length of a rectangle is 5 inches more than its width; its area is 374 square inches. What are its dimensions?

72. The longer leg of a right triangle is 1 inch less than twice the shorter leg, the hypotenuse is 1 inch greater than twice the shorter leg. Find the lengths of the three sides of the triangle.

73. A flower bed 18 by 30 feet is bordered by a gravel walk of uniform width. The area of the walk is three-fifths of the area of the bed. Find the width of the walk.

74. A boy has mowed a strip 6 feet wide around the edges of a rectangular lawn. The area of the mowed part is 1056 square feet, the area of the unmowed part is 1120 square feet. Find the dimensions of the lawn.

75. Find two consecutive numbers whose product is 552.

76. Find two consecutive odd numbers whose product is 323.

77. Find two consecutive even numbers whose product is 1088.

78. Find two consecutive numbers the sum of whose squares is 481.

79. A grocer paid \$180 for some crates of fruit. Four crates spoiled and had to be thrown away, but he sold the rest at an increase in price of \$1 per crate, gaining \$44. How many crates did he buy?

80. A party of persons chartered a bus for \$75. Three persons withdrew from the party and, as a result, the share of each of the others was increased by \$1.25. How many were in the original party?

81. A rectangular piece of cardboard is 4 inches longer than it is wide. A 5-inch square is cut from each corner, and the sides and ends are turned up to form an open box. The box contains 700 cubic inches. What are the dimensions of the cardboard?

82. A rectangular piece of tin is twice as long as it is wide. A pan having a capacity of 572 cubic inches is made by cutting a 2-inch square from each corner and bending up the sides and ends. Find the dimensions of the piece of tin.

83. A rod which weighs 3 pounds per foot can be made to balance at a point 3 feet from one end by attaching a weight of 20 pounds to that end. How long is the rod?

84. Two airplanes together can map a certain area in 12 hours. One of the planes alone requires 10 hours more than the other to map this area. How long would it take each plane separately to do the mapping?

85. A man rowed 6 miles upstream and back (total distance 12 miles) in 4 hours. His rate of rowing in still water is 4 miles an hour. Find the rate of the current.

86. An airplane flew 550 miles at a constant rate. If its speed had been 25 miles per hour faster the trip would have been made in 10 minutes less time. How fast did it fly?

87. A man started out to drive to a town 225 miles away. By maintaining the rate at which he started out he would have completed the trip in 5 hours. However, after driving three-fifths of the distance, he had a flat tire, which delayed him 45 minutes. He increased his speed 5 miles an hour for the remainder of the trip and reached his destination 33 minutes later than he had planned. At what rate did he start out?

88. An airplane flying west at the rate of 200 miles an hour passes over an airport at noon. A second plane flying south at the rate of 250 miles an hour passes over the same airport 24 minutes later. When are they 130 miles apart?

89. An automobile starts from the intersection of two highways

and travels north at the rate of 50 miles an hour. Four minutes later another automobile starts from the same intersection and travels east at the rate of 45 miles an hour. How long after the first car sets out will the total distance that the cars have traveled be 40 per cent greater than their distance apart?

90. A motorcycle messenger left the rear of a motorized troop 8 miles long and rode to the front of the troop, returning at once to the rear. How far did he ride, if the troop traveled 15 miles during this time and each traveled at a uniform rate?

In analytic geometry the standard form of the equation of a circle is

$$(x - h)^2 + (y - k^2) = r^2,$$

in which h and k are the coordinates of the center and r is the radius. The quantities h and k may be positive or negative, but r is restricted to positive values. By completing squares, reduce each of the following equations of circles to the foregoing form and give the coordinates of the center, also the radius:

91. $x^2 + y^2 - 6x + 4y - 3 = 0.$

SOLUTION. $x^2 - 6x + 9 + y^2 + 4y + 4 = 3 + 9 + 4,$
$(x - 3)^2 + (y + 2)^2 = 16.$

Coordinates of center are $(3, -2)$, radius $= \sqrt{16} = 4.$

92. $x^2 + y^2 - 4x - 10y - 71 = 0.$
93. $x^2 + y^2 + 2x - 48 = 0.$
94. $x^2 + y^2 + 16x + 14y - 8 = 0.$
95. $x^2 + y^2 - 6x + 2y + 6 = 0.$
96. $x^2 + y^2 - x + 3y - 2 = 0.$
97. $x^2 + y^2 - 6x - 8y = 0.$
98. $x^2 + y^2 - 9x + 6y + 17 = 0.$
99. $x^2 + y^2 + 10x - 2y + 25 = 0.$
100. $x^2 + y^2 - 4x + 8y - 44 = 0.$
101. $x^2 + y^2 + 22x + 30y - 230 = 0.$

In analytic geometry one of the standard forms of the equation of an ellipse with center at the point (h,k) and axes parallel to the coordinate axes is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1.$$

Reduce each of the following equations of ellipses to this standard form and give the coordinates of the center and the values of a and b:

102. $4x^2 + 9y^2 - 8x + 90y + 193 = 0.$

SOLUTION. Arrange as follows:

$$4(x^2 - 2x) + 9(y^2 + 10y) = -193.$$

Complete the squares within the parentheses and compensate for this by adding the proper amounts to the right-hand side:

$$4(x^2 - 2x + 1) + 9(y^2 + 10y + 25) = -193 + 4 \cdot 1 + 9 \cdot 25,$$
$$4(x-1)^2 + 9(y+5)^2 = 36.$$

Divide both sides by 36:

$$\frac{(x-1)^2}{9} + \frac{(y+2)^2}{4} = 1.$$

Coordinates of center are $(1,-2)$; $a = \sqrt{9} = 3$, $b = \sqrt{4} = 2$.

103. $9x^2 + 16y^2 + 54x + 64y + 1 = 0.$
104. $x^2 + 4y^2 + 6x - 16y + 9 = 0.$
105. $16x^2 + 25y^2 + 64x + 50y - 311 = 0.$
106. $2x^2 + 3y^2 - 8x - 18y + 29 = 0.$
107. $9x^2 + 25y^2 + 90x - 150y - 225 = 0.$
108. $16x^2 + 25y^2 - 160x + 50y - 1175 = 0.$
109. $5x^2 + 27y^2 - 10x + 108y - 22 = 0.$
110. $9x^2 + 36y^2 - 6x - 48y - 307 = 0.$

111. $16x^2 + 81y^2 + 192x + 324y + 864 = 0.$
112. $5x^2 + 7y^2 - 80x - 84y + 537 = 0.$

In analytic geometry one of the standard forms of the equation of a hyperbola with center at the point (h,k) and axes parallel to the coordinate axes is

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1.$$

Reduce each of the following equations of hyperbolas to this standard form and give the coordinates of the center and the values of a and b:

113. $25x^2 - 9y^2 - 100x + 72y - 269 = 0.$

SOLUTION. Arrange as follows:

$$25(x^2 - 4x) - 9(y^2 - 8y) = 269.$$

Complete the squares within the parentheses and compensate for this by adding and subtracting the proper amounts on the right-hand side:

$$25(x^2 - 4x + 4) - 9(y^2 - 8y + 16) = 269 + 25 \cdot 4 - 9 \cdot 16,$$
$$25(x-2)^2 - 9(y-4)^2 = 225.$$

Divide both sides by 225:

$$\frac{(x-2)^2}{9} - \frac{(y-4)^2}{25} = 1.$$

Coordinates of center are (2,4); $a = \sqrt{9} = 3$, $b = \sqrt{25} = 5$.

114. $4x^2 - 9y^2 - 8x - 36y - 68 = 0.$
115. $25x^2 - 16y^2 - 150x - 128y - 431 = 0.$
116. $x^2 - 36y^2 - 2x - 72y - 71 = 0.$
117. $9x^2 - 144y^2 + 36x + 576y - 1836 = 0.$
118. $x^2 - 9y^2 - 8x + 90y - 234 = 0.$

119. $x^2 - y^2 - 14x + 10y + 20 = 0.$
120. $4x^2 - 9y^2 + 24x - 36y - 1 = 0.$
121. $3x^2 - 4y^2 + 6x + 16y - 37 = 0.$
122. $16x^2 - 36y^2 - 48x - 36y - 549 = 0.$
123. $4x^2 - 18y^2 + 4x - 108y - 185 = 0.$

In integral calculus it is frequently desirable to change an expression of the form $\sqrt{ax^2 + bx + c}$, $a > 0$, into the form $\sqrt{a}\sqrt{(x - h)^2 + k}$ or an expression of the form $\sqrt{c + bx - ax^2}$, $a > 0$, into the form $\sqrt{a}\sqrt{k - (x - h)^2}$. Transform each of the following expressions into the appropriate one of these forms:

124. $\sqrt{2x^2 - 8x + 14}.$
125. $\sqrt{5x^2 - 40x + 60}.$
126. $\sqrt{3x^2 - 6x + 7}.$
127. $\sqrt{4x^2 + 40x + 91}.$
128. $\sqrt{5x^2 + 2x - 3}.$
129. $\sqrt{2x^2 + 2x + 1}.$
130. $\sqrt{7x^2 + 49x}.$
131. $\sqrt{3x^2 - x - 2}.$
132. $\sqrt{4x^2 + 2x - 1}.$
133. $\sqrt{3x^2 + 4x}.$
134. $\sqrt{3 + 6x - x^2}.$
135. $\sqrt{10 + 4x - 2x^2}.$
136. $\sqrt{15 - 10x - 5x^2}.$
137. $\sqrt{3 - 2x - 2x^2}.$
138. $\sqrt{1 + 18x - 3x^2}.$
139. $\sqrt{6x - x^2}.$
140. $\sqrt{14x - 5x^2}.$
141. $\sqrt{2 - x - 2x^2}.$
142. $\sqrt{1 - 7x - 3x^2}.$
143. $\sqrt{3 + 6x - 4x^2}.$

50. Equations in quadratic form.

An equation is **in quadratic form** if it is a quadratic in some function of the original unknown. Thus,

$$x^4 + 5x^2 - 36 = 0 \text{ is a quadratic in } x^2,$$
$$2x + 3\sqrt{x} - 5 = 0 \text{ is a quadratic in } \sqrt{x}.$$

Example 1.

Solve $$x^4 + x^2 - 20 = 0.$$

SOLUTION. Factor:

$$(x^2 - 4)(x^2 + 5) = 0.$$

Set each factor separately equal to zero, getting

$$x^2 = 4, \qquad x = \pm 2.$$
$$x^2 = -5, \qquad x = \pm\sqrt{-5} = \pm i\sqrt{5}.$$

Example 2.

Solve $$\frac{x^2 - 5}{x^2 + 3} + \frac{x^2 + 3}{x^2 - 5} = -2.$$

SOLUTION. Let

$$\frac{x^2 - 5}{x^2 + 3} = y.$$

Then $$y + \frac{1}{y} = -2,$$
$$y^2 + 1 = -2y,$$
$$y^2 + 2y + 1 = 0,$$
$$y = -1, \quad \text{or} \quad \frac{x^2 - 5}{x^2 + 3} = -1.$$
$$x^2 = 1,$$
$$x = \pm 1.$$

Note that the two fractions in the equation are reciprocals, and that we could equally well have represented the second, instead of the first, by y.

EXERCISES VI. B

Solve the following equations. If an equation involves radicals or fractional exponents all answers must be tested by substitution into the original equation.

1. $x^4 - 13x^2 + 36 = 0.$ **2.** $x^4 - 19x^2 + 48 = 0.$
3. $18x^4 - 5x^2 - 48 = 0.$ **4.** $x^{-4} + 21x^{-2} - 100 = 0.$
5. $x^6 + 7x^3 - 8 = 0.$ **6.** $x^{-6} - 19x^{-3} - 216 = 0.$
7. $10x - 9\sqrt{x} + 2 = 0.$ **8.** $6x - 11\sqrt{x} - 35 = 0.$

9. $x^{2/3} - 5x^{1/3} + 6 = 0.$
10. $8x^{2/3} + 14x^{1/3} - 15 = 0.$
11. $2x^{1/4} - x^{1/2} + 24 = 0.$
12. $x^{-2/3} + 5x^{-1/3} - 36 = 0.$
13. $x^{-4/3} - 17x^{-2/3} + 16 = 0.$
14. $x^{2/3} - 4x^{1/3} + 13 = 0.$
15. $2\sqrt[3]{x} - 7\sqrt[6]{x} + 3 = 0.$
16. $9x^{-1/2} - 37x^{-1/4} + 4 = 0.$

17. $\left(\frac{2x}{3x-1}\right)^2 - \frac{2x}{3x-1} - 2 = 0.$

18. $\frac{2x}{3x-1} - \sqrt{\frac{2x}{3x-1}} - 2 = 0.$

19. $\left(x - \frac{2}{x}\right)^2 - 3x + \frac{6}{x} = 4.$

20. $\sqrt{\frac{x+3}{x-3}} + \sqrt{\frac{x-3}{x+3}} = \frac{5}{2}.$

21. $\sqrt{\frac{x+5}{x+2}} + 6\sqrt{\frac{x+2}{x+5}} = 5.$

22. $\frac{x^2+4}{x^2-7} + \frac{x^2-7}{x^2+4} = \frac{26}{5}.$

23. $(x^2 - 3x + 5)^2 - 3x^2 + 9x - 43 = 0.$

24. $x^2 - 5x - 6 - 5\sqrt{x^2 - 5x} = 0.$

25. $2x^2 - \sqrt{x^2 - 2x + 6} = 4x + 3.$

26. $3x^2 - 10x - 5 = \frac{24}{3x^2 - 10x}.$

27. $x^2 + 5x - 2 = -\frac{8}{x^2 + 5x + 4}.$

28. The so-called effective area of the cross section of a chimney is given by the formula $E = A - 0.6\sqrt{A}$, in which A is the actual area. What must be the actual area of a chimney if it is to have an effective area of 22 square feet?

29. The distance d, in feet, that an object will fall in t seconds is given by the formula $d = \frac{1}{2}gt^2$, in which $g = 32$ approximately. A stone is dropped into the shaft of an abandoned mine and 6 seconds later the sound of its striking the bottom of the shaft is heard. Assuming that sound travels 1190 feet per second, calculate the depth of the shaft.

51. Equations involving radicals.

An equation involving radicals can sometimes be solved by the method of the preceding section. Another method of solution is illustrated in the following examples.

Example 1.

Solve
$$x + \sqrt{x} - 6 = 0.$$

SOLUTION.

Transpose $\sqrt{x}$:
$$x - 6 = -\sqrt{x}.$$

Square:
$$x^2 - 12x + 36 = x,$$
$$x^2 - 13x + 36 = 0,$$
$$(x - 4)(x - 9) = 0,$$
$$x = 4,\ 9.$$

The value 4 checks, since

$$4 + \sqrt{4} - 6 = 4 + 2 - 6 = 0.$$

The value 9 however does not check, since

$$9 + \sqrt{9} - 6 = 9 + 3 - 6 \neq 0.$$

This is not because we have made an error, but because by squaring both sides of the equation, we have introduced an extraneous root. (See section 32.) The value 9 must be discarded as a possible root, since it does not satisfy the original equation. Thus, the only root of the equation is $x = 4$.

In dealing with equations involving radicals, all solutions obtained must be tested, as some may be extraneous. In this connection it should be recalled that $\sqrt{x}$, when x is positive, means the positive square root of x only.

Example 1 can be solved by the method of the preceding section, as follows:

$$x + \sqrt{x} - 6 = 0,$$
$$(\sqrt{x} + 3)(\sqrt{x} - 2) = 0.$$

Since $\sqrt{x}$ cannot be negative, $\sqrt{x} + 3$ cannot be zero; hence

$$\sqrt{x} - 2 = 0, \qquad \sqrt{x} = 2, \qquad x = 4.$$

This method is preferable here.

Example 2.

Solve $$\sqrt{3x + 7} + \sqrt{x + 1} - 2 = 0.$$

Solution. Transpose, keeping one radical alone on one side of the equation. (If there is any difference in the radicals always *isolate the most complicated.*)

$$\sqrt{3x + 7} = 2 - \sqrt{x + 1}.$$

Square: $$3x + 7 = 4 - 4\sqrt{x + 1} + x + 1.$$

Collect terms, keeping the radical alone on one side:

$$4\sqrt{x + 1} = -2x - 2.$$

Divide by 2: $$2\sqrt{x + 1} = -x - 1.$$

Square again: $$4x + 4 = x^2 + 2x + 1,$$

$$x^2 - 2x - 3 = 0,$$

$$(x - 3)(x + 1) = 0,$$

$$x = 3, \ -1.$$

The value 3 does not satisfy the equation and must be discarded as a root.

Example 3.

Solve $$\sqrt{x + 1} + 3 = 0.$$

Discussion. We could proceed as above, obtaining values for x. This is useless, however, for both terms are positive, namely, the positive root of $x + 1$ and the positive number 3, and it is impossible for their sum to equal zero.

Example 4.

Solve $$6x^2 + 10x + \sqrt{3x^2 + 5x + 1} = -1.$$

SOLUTION BY REDUCING TO QUADRATIC FORM. Add 2 to both sides:

$$6x^2 + 10x + 2 + \sqrt{3x^2 + 5x + 1} = 1,$$
$$2(3x^2 + 5x + 1) + \sqrt{3x^2 + 5x + 1} = 1.$$

Let $\sqrt{3x^2 + 5x + 1} = y$:
$$2y^2 + y - 1 = 0,$$
$$(2y - 1)(y + 1) = 0,$$
$$y = \frac{1}{2}, -1.$$

$$\sqrt{3x^2 + 5x + 1} = \frac{1}{2} \text{ or } -1.$$

Since a positive radical cannot equal -1, we need only to consider the equation in which the right side is $\frac{1}{2}$. Squaring this, we get

$$3x^2 + 5x + 1 = \frac{1}{4},$$
$$12x^2 + 20x + 3 = 0,$$
$$x = -\frac{1}{6}, -\frac{3}{2}.$$

Both of these values satisfy the original equation.

EXERCISES VI. C

Solve:

1. $\sqrt{x - 5} - 6 = 0.$ **2.** $\sqrt{2x + 3} - 4 = 0.$

3. $\sqrt{34x - 15} - 4x = 0.$ **4.** $\sqrt{3x - 2} + 8 = 2x.$

5. $\sqrt{2x + 5} - 6x = 5.$ **6.** $\sqrt{x - 5} + 2\sqrt{x - 3} = 0.$

7. $\sqrt{2x + 5} - \sqrt{x - 6} = 3.$

8. $2\sqrt{2x + 1} - \sqrt{3x + 4} = 2.$

9. $2\sqrt{x - 9} - \sqrt{x - 16} = \sqrt{x}.$

10. $\sqrt{5x + 2} + \sqrt{x + 2} = 2\sqrt{3x}.$

11. $\sqrt{2x + 1} - \sqrt{4x - 5} + \sqrt{6x - 8} = 0.$

12. $\sqrt{2x - 3} - 4 = \dfrac{5}{\sqrt{2x - 3}}.$

13. $\dfrac{\sqrt{4x+1}}{3} + \dfrac{3}{\sqrt{4x+1}} = 2.$

14. $\sqrt{x^2 - 10x + 34} + \sqrt{x^2 + 10x + 34} = 4\sqrt{10}.$

15. $\dfrac{\sqrt{2+x} + \sqrt{2-x}}{\sqrt{2+x} - \sqrt{2-x}} = 3.$

16. $\dfrac{2 + \sqrt{x+3}}{2 - \sqrt{x+3}} = \dfrac{3 + \sqrt{x+2}}{3 - \sqrt{x+2}}.$

17. $\sqrt{x^2 + 2x - 2\sqrt{x^2 + 2x - 6}} - x = 0.$

18. $\sqrt[3]{x - 1 + \sqrt{x+3}} - 2 = 0.$

19. $\sqrt[3]{3x^3 - 11} + 2\sqrt{x} = 0.$

20. One leg of a right triangle is 24 inches. If the other leg were lengthened by 25 inches the hypotenuse would be lengthened by 15 inches. Find the length of the hypotenuse.

21. Two straight wires are attached at a point 24 feet above the base of a vertical flagpole standing on level ground. One wire is 1 foot longer than the other and reaches the ground at a point 3 feet farther from the base. Find the length of each.

22. The radius of the base of each of two right circular cones is 8 inches. The altitude of the taller cone is 9 inches more than that of the shorter cone, and its slant height is 7 inches more than that of the shorter cone. Find the altitude of each.

23. A lighthouse is located 5 miles out from a straight coast. The lighthouse keeper has a boat which can travel at the rate of 10 miles an hour. At some distance along the coast is a port. If he goes by boat directly to this port it takes him half an hour longer than if he goes by boat to the nearest point on the coast and then proceeds by automobile, at the rate of 40 miles an hour, to the port. How far is it from the lighthouse to the port?

52. Character of the roots.

It will be recalled that the two roots of the quadratic equation $ax^2 + bx + c = 0$ are

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \qquad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}. \qquad (1)$$

The expression

$$b^2 - 4ac,$$

appearing under the radical sign, is called the **discriminant** of the equation $ax^2 + bx + c = 0$.

If a, b, c are real numbers, then the discriminant gives us the following information regarding the character of the roots:

$$\textit{If } b^2 - 4ac \textit{ is } \begin{Bmatrix} + \\ 0 \\ - \end{Bmatrix} \textit{ the roots are } \begin{cases} \textit{real and unequal.} \\ \textit{real and equal.} \\ \textit{imaginary.} \end{cases}$$

If a, b, c are rational,* the roots are rational when $b^2 - 4ac$ is a perfect square (including zero); otherwise they are irrational.

If the roots are imaginary, they are of the form $p + qi$ and $p - qi$, that is, they are conjugate imaginary quantities.

Two equal roots are often called a **double root.**

The examples tabulated on page 95 illustrate the use of the discriminant in determining, without solving the quadratic equation, the character of its roots. The final column is unnecessary, but is included here for purposes of verification.

If the constant term in a quadratic equation is missing, one root of the equation is zero. Thus, the equation $2x^2 - 3x = 0$ has one root 0. (The other root is $3/2$.)

If both the constant term and the first-degree term are missing, both roots of the quadratic equation are zero; this is, of course, a trivial equation. Thus, the roots of $2x^2 = 0$ are 0 and 0.

* See p. 39, footnote.

Equation	Discriminant $b^2 - 4ac$	Roots are	Values of roots
$3x^2 - 4x - 7 = 0$	$(-4)^2 - 4 \cdot 3 \cdot (-7)$ $= 16 + 84 = 100$	real and unequal (also rational, since 100 is a perfect square)	$\frac{7}{3}, -1$
$2x^2 + 2x + 5 = 0$	$2^2 - 4 \cdot 2 \cdot 5$ $= 4 - 40 = -36$	imaginary	$-\frac{1}{2} + \frac{3}{2}i,$ $-\frac{1}{2} - \frac{3}{2}i$
$4x^2 + 12x + 9 = 0$	$(12)^2 - 4 \cdot 4 \cdot 9$ $= 144 - 144 = 0$	real and equal (also rational, of course)	$-\frac{3}{2}, -\frac{3}{2}$

If the first-degree term is missing, but the constant term is present, the roots are numerically equal but of opposite sign. Thus, the roots of $2x^2 - 3 = 0$ are $x = \sqrt{3/2}$ and $x = -\sqrt{3/2}$; the roots of $x^2 + 4 = 0$ are $x = 2i$ and $x = -2i$.

EXERCISES VI. D

Determine the character of the roots of the following equations without solving:

1. $x^2 - 4x - 16 = 0.$
2. $x^2 + 12x + 36 = 0.$
3. $x^2 + 2x + 2 = 0.$
4. $x^2 + 3x + 1 = 0.$
5. $2x^2 + 5x - 4 = 0.$
6. $2x^2 + 5x + 6 = 0.$
7. $4x^2 - 28x + 49 = 0.$
8. $2x^2 + 5x + 3 = 0.$
9. $4x^2 - 49 = 0.$
10. $4x^2 - 49x = 0.$
11. $4x^2 + 49 = 0.$
12. $4x^2 + 49x + 4 = 0.$
13. $5x^2 = 4x + 3.$
14. $2x^2 = 5x - 2.$
15. $1776x^2 + 1839x + 1896 = 0.$
16. $837x^2 - 1552x + 2814 = 0.$

Determine the values of k that will make the roots of the following equations equal:

17. $x^2 + kx + k = 0.$

SOLUTION. $b^2 - 4ac = k^2 - 4k = 0,$

$$k = 0, 4.$$

If $k = 0$ the equation reduces to $x^2 = 0$, which has both roots zero. If $k = 4$ the equation reduces to $x^2 + 4x + 4 = 0$, both of whose roots are $x = -2$.

18. $kx^2 + 2x - 3 = 0.$ **19.** $2x^2 - 3x + 2 = k.$
20. $9x^2 + 9x + 4 = kx.$ **21.** $x^2 - 5x + 1 = kx^2.$
22. $(x + 1)^2k - (k + 1)^2x = 0.$
23. $2(x + k)^2 = k(x + 2).$
24. $x^2 + k^2 + kx + 6x + 12 = 0.$
25. $4x^2 + k^2 - 4kx - 16x + 16k - 8 = 0.$
26. $(x - 1)(x - 2) - (x - 1)(k - 2) + (x - 2)(k - 1) = 0.$
27. $x^2 + 4k^2 + 6x - 4kx = 12k - 9.$

Determine k in each of the following equations so that one root of the equation will be zero:

SUGGESTION. Set the constant term equal to zero.

28. $2x^2 - 3x + 2 = k.$ **29.** $(x + 1)^2 - (k + 2)^2 = 0.$
30. $2(x + k)^2 = k(x + 3).$ **31.** $9x^2 + 9x + 4 = kx.$
32. $x^2 + k^2 + 6(x^2 - 1) - 5(x + k) = 0.$
33. $(x - 1)(x - 2) - (x - 1)(k - 2) + (x - 2)(k - 1) = 0.$
34. $(kx - 3)^2 + 3(x - 3)^2 - (k - 3)^2 = 0.$
35. $x^2 - k^2 + kx - 6x + 12 = 0.$

Determine k so that the roots of the following equations will be numerically equal but of opposite sign:

36. $x^2 + 4k^2 - 4kx + 3x - 4k + 5 = 0.$
37. $(x - 1)(x - 2) - (x - 1)(k - 2) + (x - 2)(k - 1) = 0.$
38. $(kx - 3)^2 + 3(x - 3)^2 - (k - 3)^2 = 0.$
39. $(kx - 2)^2 = [(k - 2)(x - 2)]^2.$
40. $2(x + k)^2 = k(x + 2)^2.$
41. $\frac{kx^2}{3} + \frac{3x}{k} - \frac{kx}{3} - \frac{3}{k} = 0.$
42. $\frac{x^2 - 1}{k^2 - 1} = \frac{x + k}{k - 1} + \frac{x - k}{k + 1}.$

Determine the character of the roots of the following equations. Note that some of the coefficients are irrational numbers and that it cannot be concluded that the roots are rational if $b^2 - 4ac$ is a perfect square.

43. $x^2 + 5x + \sqrt{3} = 0$. **44.** $x^2 + \sqrt{5} \cdot x + \sqrt{3} = 0$.
45. $2x^2 - 2\sqrt{10} \cdot x + 5 = 0$. **46.** $3x^2 + 3\sqrt{3} \cdot x + \sqrt{5} = 0$.
47. $7x^2 + 3\sqrt{6} \cdot x + 2 = 0$. **48.** $7x^2 - 2\sqrt{21} \cdot x + 3 = 0$.

53. Sum and product of the roots.

If we write r for the radical $\sqrt{b^2 - 4ac}$, then equations (1) of section 52 reduce to

$$x_1 = \frac{-b + r}{2a}, \qquad x_2 = \frac{-b - r}{2a}.$$

Then,

$$x_1 + x_2 = \frac{-b + r}{2a} + \frac{-b - r}{2a} = -\frac{2b}{2a} = -\frac{b}{a},$$

$$x_1 x_2 = \frac{b^2 - r^2}{4a^2} = \frac{b^2 - (b^2 - 4ac)}{4a^2} = \frac{4ac}{4a^2} = \frac{c}{a}.$$

That is, *in the quadratic equation* $ax^2 + bx + c = 0$, *the sum of the roots is* $-\frac{b}{a}$ *and the product of the roots is* $\frac{c}{a}$.

It is perhaps simpler to divide the equation through by the coefficient of x^2, reducing it to the form $x^2 + px + q = 0$. Then we can state that *in the quadratic equation* $x^2 + px + q = 0$, *the sum of the roots is* $-p$ *and the product of the roots is* q.

Example.

Find the sum and the product of the roots of the equation

$$3x^2 - 4x - 7 = 0.$$

SOLUTION. Divide by 3:

$$x^2 - \frac{4}{3}x - \frac{7}{3} = 0$$

Sum of roots $= -\left(-\frac{4}{3}\right) = \frac{4}{3}$.

Product of roots $= -\frac{7}{3}$.

These results can easily be checked in the present example by solving. The roots are $\frac{7}{3}$ and -1.

54. Formation of an equation with given roots.

If x_1 and x_2 are the roots of a quadratic equation, then the equation can be written $(x - x_1)(x - x_2) = 0$.

Example 1.

Find an equation whose roots are 2 and -3.

SOLUTION. *Method 1.* $(x - 2)(x + 3) = 0$,

$$x^2 + x - 6 = 0.$$

Method 2.

Sum of roots $= 2 - 3 = -1 = -p$, $p = 1$.
Product of roots $= 2(-3) = -6 = q$.

$$x^2 + px + q \equiv x^2 + x - 6 = 0.$$

Example 2.

Find an equation whose roots are $\frac{2}{3}$, $-\frac{1}{2}$.

SOLUTION. $$\left(x - \frac{2}{3}\right)\left(x + \frac{1}{2}\right) = 0.$$

Multiply by $3 \cdot 2$ to clear of fractions:

$$3\left(x - \frac{2}{3}\right)2\left(x + \frac{1}{2}\right) = 0,$$

$$(3x - 2)(2x + 1) = 0,$$

$$6x^2 - x - 2 = 0.$$

Example 3.

Find an equation whose roots are 2, −3, 7.

SOLUTION.
$$(x - 2)(x + 3)(x - 7) = 0,$$
$$x^3 - 6x^2 - 13x + 42 = 0.$$

Note that this is not a quadratic equation.

Example 4.

Find an equation whose roots are x_1 and x_2.

SOLUTION.
$$(x - x_1)(x - x_2) = 0,$$
$$x^2 - (x_1 + x_2)x + x_1x_2 = 0.$$

Compare this result with section 53.

55. Factoring by solving a quadratic.

A quadratic expression with rational coefficients can be factored into rational factors if and only if its discriminant is a perfect square.

Example.

Factor $3x^2 - 4x - 7$.

SOLUTION. $b^2 - 4ac = 100$. Solve the equation

$$3x^2 - 4x - 7 = 0.$$

$$x = \frac{4 \pm \sqrt{16 + 84}}{6} = \frac{4 \pm 10}{6} = \frac{7}{3}, -1.$$

The factors of the original expression are

$$3\left(x - \frac{7}{3}\right)(x + 1), \quad \text{or} \quad (3x - 7)(x + 1).$$

In general, the factors of $ax^2 + bx + c$ are

$$a(x - x_1)(x - x_2),$$

where x_1 and x_2 are given by (1) of section 52.

EXERCISES VI. E

Find, without solving, the sum and the product of the roots of the following equations:

1. $x^2 + 5x + 3 = 0.$
2. $3x^2 - 7x + 4 = 0.$
3. $4x^2 - 2x - 1 = 0.$
4. $5x^2 + 6x + 7 = 0.$
5. $8 - 9x - 6x^2 = 0.$
6. $8x^2 + 3x = 4.$
7. $(3x + 2)^2 = 4.$
8. $25x^2 - 6 = 0.$
9. $25x^2 - 6x = 0.$
10. $\frac{1}{2}x^2 + \frac{1}{3}x + \frac{1}{4} = 0.$
11. $kx^2 - k^2x - 2x(x - 1) + k(x - 1) - k = 0.$

Determine the value of k in the following equations:

12. $x^2 + kx + 18 = 0$, given that one root is twice the other.

SOLUTION. Let r = one root; then $2r$ is the other.

$$\left.\begin{aligned} r + 2r &= -k, \\ r \cdot 2r &= 18; \end{aligned}\right\} \quad \text{or} \quad \begin{cases} 3r = -k, \\ r^2 = 9, \quad r = \pm 3. \end{cases}$$
$$k = -3r = \mp 9.$$

13. $x^2 + kx + 5 = 0$, given that one root is 4.

SUGGESTION FOR ALTERNATIVE METHOD OF SOLUTION. Since 4 is a root it may be substituted for x and the resulting equation solved for k.

14. $3x^2 + kx + k + 6 = 0$, given that one root is 5.
15. $6x^2 - 7x - k = 0$, given that one root is $-\frac{2}{3}$.
16. $x^2 - kx - 24 = 0$, given that the difference between the roots is 11.
17. $6x^2 - 17x + k = 0$, given that the difference between the roots is $\frac{1}{6}$.
18. $x^2 + kx + 1 = 0$, given that one root is 4 times the other.
19. $3x^2 + 8x + k = 0$, given that one root is 3 times the other.
20. $4x^2 + kx - 7 = 0$, given that the quotient of the roots is $-\frac{9}{28}$.
21. $ax^2 + bx + k = 0$, given that one root is the reciprocal of the other.
22. Let r and s be the roots of the equation $ax^2 + bx + c = 0$. Find a quadratic equation whose roots are r/s and s/r.

Find quadratic equations whose roots are

23. $5, 7.$ **24.** $-3, 6.$ **25.** $4, 4.$
26. $-8, 0.$ **27.** $\frac{5}{8}, -\frac{3}{4}.$ **28.** $\pm\frac{6}{7}.$
29. $\pm\sqrt{3}.$ **30.** $\pm 5i.$ **31.** $\pm i\sqrt{5}.$
32. $1 \pm \sqrt{3}.$ **33.** $-3 \pm \sqrt{7}.$ **34.** $2 \pm 3i.$
35. $2 \pm i\sqrt{3}.$ **36.** $\frac{1}{2}(7 \pm 3\sqrt{2}).$ **37.** $\frac{1}{3}(5 \pm i\sqrt{3}).$
38. $\dfrac{1}{2} \pm \dfrac{1}{3}\sqrt{3}.$ **39.** $\dfrac{1}{3} \pm \dfrac{\sqrt{2}}{5}i.$ **40.** $-\dfrac{2}{3} \pm \dfrac{3\sqrt{2}}{4}i.$
41. $\sqrt{2} \pm \sqrt{3}.$ **42.** $\sqrt{2} \pm 3i.$ **43.** $\sqrt{2} \pm i\sqrt{3}.$
44. $5, -1, 6.$ **45.** $3, -\frac{1}{3}, -\frac{5}{6}.$ **46.** $5, \pm 4.$

EXERCISES VI. F

Factor the following expressions by first solving the corresponding equations:

1. $24x^2 - 1490x - 625.$ **2.** $81x^2 - 1515x + 56.$
3. $64x^2 - 66x - 49.$ **4.** $54x^2 + 219x + 200.$

Find the irrational or the imaginary factors of the following expressions:

5. $x^2 - 2x - 1.$

Solution. Set $x^2 - 2x - 1 = 0$ and solve, getting

$$x_1 = 1 + \sqrt{2}, \qquad x_2 = 1 - \sqrt{2}.$$

The factors are

$$[x - (1 + \sqrt{2})][x - (1 - \sqrt{2})], \quad \text{or}$$
$$(x - 1 - \sqrt{2})(x - 1 + \sqrt{2}).$$

6. $x^2 + 25.$ **7.** $x^2 + 5.$
8. $x^2 - 12x + 24.$ **9.** $x^2 + 6x + 13.$
10. $4x^2 - 20x + 34.$ **11.** $3x^2 - 4x - 1.$
12. $4x^2 - 24x + 41.$ **13.** $x^2 - 2\sqrt{2}\cdot x - 1.$
14. $x^2 - 2\sqrt{5}\cdot x + 14.$ **15.** $x^2 - 2\sqrt{5}\cdot x + 8.$

56. Graphic representation of a quadratic function.

If we have a quadratic function such as $x^2 - 2x - 3$, we can set $y = x^2 - 2x - 3$, find values of y corresponding to assigned values of x (see accompanying table), and plot. The graph of this function is shown in Fig. 9. The curve is called a **parabola.** (The graph of any quadratic function is a parabola.) The points at which the curve crosses the x-axis, namely, $x = -1$ and $x = 3$, are the real **zeros** of the quadratic function, or the real roots of the corresponding equation, $x^2 - 2x - 3 = 0$.

$y = x^2 - 2x - 3$

x	y
-2	5
-1	0
0	-3
1	-4
2	-3
3	0
4	5

The graph of the function $y = x^2 - 2x + 1$ is shown in Fig. 10. The equation found by setting the function equal to zero has equal roots (i.e., a double root) $x = 1$, and it will be observed that the curve merely touches the x-axis at the point $x = 1$.

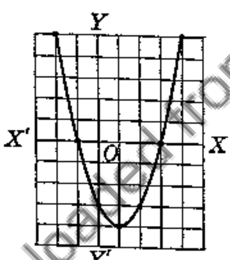


FIG. 9. Graph of quadratic function $x^2 - 2x - 3$. Roots of corresponding equation real and unequal.

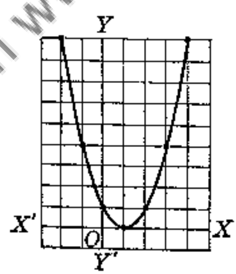


FIG. 10. Graph of quadratic function $x^2 - 2x + 1$. Roots of corresponding equation real and equal.

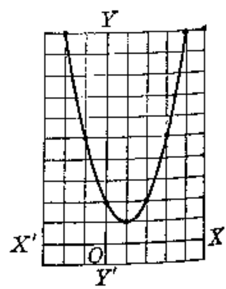


FIG. 11. Graph of quadratic function $x^2 - 2x + 2$. Roots of corresponding equation imaginary.

Fig. 11 shows the graph of $y = x^2 - 2x + 2$. The corresponding equation has imaginary roots $x = 1 \pm i$, and the curve does not meet the x-axis.

In general, *the graph of a quadratic function crosses the x-axis in two distinct points if the roots of the equation obtained by setting the function equal to zero are real and* *unequal, touches the x-axis if they are real and equal, and does not meet the x-axis at all if they are imaginary.*

57. Maximum or minimum value of a quadratic function.

The lowest point on the curve representing the quadratic function corresponds to the **minimum,** or least, value of the function. If the curve is inverted from the positions shown in Figs. 9, 10, 11, as will be the case if the coefficient of x^2 is negative, the highest point on the curve corresponds to the **maximum,** or greatest, value of the function. A method of finding the maximum or minimum value of a quadratic function by completing the square is illustrated in the examples below.

Example 1.

Find the minimum value of $3x^2 - 2x + 4$.

SOLUTION.
$$y = 3\left(x^2 - \frac{2}{3}x\right) + 4$$
$$= 3\left(x^2 - \frac{2}{3}x + \frac{1}{9}\right) + 4 - \frac{1}{3}$$
$$= 3\left(x - \frac{1}{3}\right)^2 + \frac{11}{3}.$$

The term $3(x - \frac{1}{3})^2$ is always positive or zero; consequently y can never be less than $\frac{11}{3}$. It is equal to this minimum value of $\frac{11}{3}$ when $x = \frac{1}{3}$.

Example 2.

Find the maximum value of $-5x^2 - 4x + 7$.

SOLUTION.
$$y = -5x^2 - 4x + 7$$
$$= -5\left(x^2 + \frac{4}{5}x\right) + 7$$
$$= -5\left(x^2 + \frac{4}{5}x + \frac{4}{25}\right) + 7 + \frac{4}{5}$$
$$= -5\left(x + \frac{2}{5}\right)^2 + \frac{39}{5}.$$

The term $-5(x + \frac{2}{5})^2$ is always negative or zero, and the maximum value of y is $\frac{39}{5}$; y is equal to $\frac{39}{5}$ when $x = -\frac{2}{5}$.

EXERCISES VI. G

Draw the graphs of the following functions:

1. $x^2 - 4$.
2. $9 - x^2$.
3. $x^2 - 4x$.
4. $x^2 + 4x$.
5. $4x - x^2$.
6. $x^2 - x + 1$.
7. $\frac{1}{2}x^2 + 1$.
8. $x^2 - 4x + 4$.
9. $x^2 - x - 6$.
10. $x^2 - 5x + 6$.
11. $2x^2 - 3x - 6$.
12. $12 + 6x - x^2$.

Find the maximum or the minimum value of each of the following functions, and the corresponding value of x. State whether maximum or minimum.

13. $x^2 - 4x + 4$.
14. $x^2 - 7x + 10$.
15. $x^2 + 5x - 3$.
16. $5 - 4x - x^2$.
17. $8 - 2x - x^2$.
18. $8 - 2x + x^2$.
19. $2x^2 - 6x - 3$.
20. $3x^2 + 2x - 6$.
21. $4x - x^2$.
22. $4x + x^2$.
23. $x^2 - x + 1$.
24. $3 - 5x - 2x^2$.
25. $\frac{1}{2}x^2 + x - 2$.
26. $\frac{1}{3}x^2 - 3x - 1$.
27. $\frac{1}{2}x^2 - \frac{1}{3}x + \frac{1}{6}$.
28. $\frac{1}{3}x^2 + \frac{1}{2}x + \frac{1}{4}$.
29. $x^2 + \sqrt{2} \cdot x + 1$.
30. $\sqrt{2} \cdot x^2 - \sqrt{3} \cdot x + 6$.

31. Find two numbers whose sum is 18 and whose product is a maximum.

32. A stone wall of indefinite length is available for one side of a rectangular enclosure, which is to be fenced in on the other three sides. Find the maximum area that can be enclosed by 100 yards of fence.

33. Prove that the rectangle having a maximum area with a fixed perimeter is a square.

34. A motion picture theater, which had a daily average of 1800 paid admissions, increased the admission price from 50 cents to 55 cents. This resulted in a decrease of 100 in the daily average of paid admissions. Assuming that for each 5-cent increase in price the number of daily paid admissions will decrease 100, find the price which will afford maximum receipts.

35. A merchant can sell 500 articles per day at a price of 60 cents apiece. He can sell 10 articles less per day for each penny that he adds to the price. If he pays 50 cents apiece wholesale for the articles, what retail price will give him the greatest profit?

36. If an object is thrown into the air with a velocity v_0, its distance d, in feet, above the point from which it was thrown, at the end of t seconds, is given by the formula

$$d = v_0t - \tfrac{1}{2}gt^2,$$

in which $g = 32$ approximately. Find the greatest height reached by an object thrown vertically into the air with a velocity of (a) 60 feet per second, (b) 64 feet per second, (c) 100 feet per second, (d) 128 feet per second, (e) 320 feet per second.

37. A man wishes to fence in a rectangular plot of ground, one side of which lies along the dividing line of his property. He has \$300 to spend for the fence, which costs \$1 per foot. He is to pay for the three sides of the enclosure which are entirely on his own land and for half of that part which is on the dividing line. What is the maximum area that he can enclose?

CHAPTER VII

Systems of Equations Involving Quadratics

58. Quadratic equations in two unknowns.

The **degree** *of a term* in two or more variables is the sum of the exponents of all the variables. (It is understood that these exponents are positive whole numbers.) Thus, $4x^3y^2$ is of degree 5, $-2x^4y^2z^5$ is of degree 11, x^2y is of degree 3. (See sections 14 and 18.)

The **degree** *of an expression or equation* is that of the term or terms of highest degree occurring in it.

A **quadratic equation** is an equation of second degree.

For the case of two unknowns, x and y, the only terms of the second degree are of the type ax^2, bxy, cy^2, and the most general quadratic equation of the second degree in x and y is

$$ax^2 + bxy + cy^2 + dx + ey + f = 0, \qquad (1)$$

terms of first degree and a constant term (degree zero) being allowable.

When we have two simultaneous equations such as (1), with all terms present, the solution would be extremely difficult; for certain special cases, however, to be considered in this chapter, the solution can always be effected.

59. One equation linear, one quadratic.

When we have two equations in two unknowns, one of the equations being linear and the other quadratic, the solution can be accomplished by substituting from the linear equation into the quadratic.

Example.

Solve:

$$3x - 2y = 5,$$
$$x^2 - xy + 2y = 7.$$

SOLUTION. From the first equation,

$$y = \frac{3x - 5}{2}.$$

Substitute in the second equation:

$$x^2 - \frac{3x^2 - 5x}{2} + 3x - 5 = 7,$$

$$2x^2 - 3x^2 + 5x + 6x - 10 = 14,$$
$$x^2 - 11x + 24 = 0,$$
$$x = 3, 8.$$

Substituting these values in the *linear* equation, we find

$$y = \frac{3x - 5}{2} = 2, \frac{19}{2}.$$

The answers must be paired as follows:

$$(x = 3, y = 2), \qquad \left(x = 8, y = \frac{19}{2}\right)$$

Both pairs of values satisfy both equations.

EXERCISES VII. A

Solve for x and y:

1. $x^2 + y^2 = 25,$
$7x + y = 25.$

2. $x^2 + y^2 - 100 = 0,$
$x + 3y + 10 = 0.$

3. $x^2 = 3y,$
$x = -3y.$

4. $y^2 = 2x,$
$y = 2x - 12.$

5. $x^2 + y^2 = 4,$
$x + y = 4.$

6. $x^2 - y^2 = 4,$
$x - y = 4.$

7. $2x - 3y = 6,$
$xy = 12.$

8. $x + y - 1 = 0,$
$3x^2 - y^2 - 23 = 0.$

9. $x^2 + y^2 - 6x - 160 = 0,$
$x + y + 4 = 0.$

10. $x^2 + y^2 - 4x + 2y - 20 = 0,$
$x - 7y + 16 = 0.$

11. $x^2 + y^2 - 4x - 6y = 2,$
$2x - y = 1.$

12. $xy + 24 = 0,$
$2x - y - 12 = 0.$

13. $x^2 - 4y = 0,$
$3x + 2y - 1 = 0.$

14. $y^2 - 4x = 0,$
$4x + 5y - 10 = 0.$

15. $3x^2 - 2xy + x - 4y = 16,$
$2x - 3y + 1 = 0.$

16. $5xy + 7y^2 - 3x + 2y + 2 = 0,$
$3x - 4y - 14 = 0.$

17. $16x^2 + 9y^2 - 32x + 36y - 92 = 0,$
$4x + 3y - 10 = 0.$

18. $9x^2 + 16y^2 + 18x + 32y - 119 = 0,$
$3x - 2y - 5 = 0.$

19. $x^2 + 2xy + 3y^2 + 5x + y - 6 = 0,$
$x - y + 3 = 0.$

20. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1,$
$\dfrac{x}{a} - \dfrac{y}{b} = 1.$

21. $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1,$
$\dfrac{x}{a} + \dfrac{y}{b} = 0.$

22. Find two numbers whose sum is 39 and whose product is 374.

23. The sum of two numbers is 26, the sum of their squares is 346. What are the numbers?

24. The hypotenuse of a right triangle is 17 inches, the difference of its two legs is 7. Find its area.

25. The perimeter of a rectangle is 98 feet, its diagonal is 41 feet. Find its dimensions.

26. The difference between the squares of the digits of a two-digit number is 45. If the digits are reversed and the resulting number subtracted from the original number the difference is also 45. What is the original number?

60. Equations linear in the squares of the unknowns.

When the unknowns occur only as squares, simultaneous quadratic equations can be solved by the methods used in solving simultaneous linear equations.

Example.

Solve the simultaneous equations

$$4x^2 + 3y^2 = 127, \quad (1)$$
$$5x^2 - 4y^2 = 89. \quad (2)$$

SOLUTION. The operations employed in effecting the solution are indicated at the left.

$4 \cdot (1)$	$16x^2 + 12y^2 = 508.$	(3)
$3 \cdot (2)$	$15x^2 - 12y^2 = 267.$	(4)
$(3) + (4)$	$31x^2 = 775.$	(5)
$(5) \div 31$	$x^2 = 25,$	(6)
	$x = \pm 5.$	
(1)	$3y^2 = 127 - 4x^2 = 127 - 100 = 27,$	
	$y = \pm 3.$	

The results should be paired as follows:

$$(x = 5, y = 3), \qquad (x = 5, y = -3),$$
$$(x = -5, y = 3), \qquad (x = -5, y = -3).$$

All four pairs of values satisfy both equations.

EXERCISES VII. B

Solve for x and y:

1. $x^2 + y^2 = 25,$
$x^2 - y^2 = 7.$

2. $x^2 + y^2 = 41,$
$2x^2 + 3y^2 = 98.$

3. $2x^2 + 3y^2 = 6,$
$3x^2 - 5y^2 = 47.$

4. $3x^2 - 5y^2 = -5,$
$4x^2 - 7y^2 = -8.$

5. $2x^2 - 3y^2 = 7,$
$3x^2 - 4y^2 = 12.$

6. $2x^2 - 5y^2 = 30,$
$3x^2 - 8y^2 = 53.$

7. $x^2 + y^2 = 25,$
$9x^2 + 25y^2 = 225.$

8. $3x^2 + 3y^2 = 4,$
$4x^2 + 4y^2 = 3.$

9. $4x^2 - 11y^2 = 7,$
$-3x^2 + 8y^2 = -4.$

10. $9x^2 + 16y^2 = 144,$
$4x^2 - 16y^2 = 64.$

11. $3x^2 + 2y^2 = 28,$
$4x^2 - 9y^2 = 0.$

12. $49x^2 - 9y^2 = 0,$
$14x^2 - 5y^2 = -833.$

13. The medians to the two legs of a right triangle are 20 and 25 respectively. Find the two legs.

61. All terms involving the unknowns of second degree.*

If all terms involving x and y are of the second degree we proceed as in the following example.

Example.

Solve:

$$2x^2 + 5xy - 10y^2 = 8, \qquad (1)$$
$$x^2 - 2xy + 3y^2 = 3. \qquad (2)$$

SOLUTION. *Method I.* Multiply (1) by 3 and (2) by 8, so as to make the constant terms the same:

$$6x^2 + 15xy - 30y^2 = 24, \qquad (3)$$
$$8x^2 - 16xy + 24y^2 = 24. \qquad (4)$$

Subtract (3) from (4):

$$2x^2 - 31xy + 54y^2 = 0. \qquad (5)$$

Solve for x in terms of y. This can be done here by factoring.

$$(x - 2y)(2x - 27y) = 0, \qquad (6)$$
$$x = 2y, \qquad x = \frac{27}{2}y. \qquad (7)$$

* Note that simultaneous quadratics of the type considered in the preceding section (viz., linear in the squares of the unknowns) are also of this type.

Substituting $x = 2y$ in (2), we get

$$3y^2 = 3, \qquad y = \pm 1. \tag{8}$$
$$x = 2y = \pm 2. \tag{9}$$

Substitute $x = \dfrac{27}{2}y$ in (2) and solve for y, getting

$$y = \pm \frac{2}{\sqrt{211}}.$$
$$x = \frac{27}{2}y = \pm \frac{27}{\sqrt{211}}.$$

The values must be paired as follows:

$$(x = 2, y = 1), \qquad (x = -2, y = -1),$$
$$\left(x = \frac{27}{\sqrt{211}}, y = \frac{2}{\sqrt{211}}\right), \qquad \left(x = -\frac{27}{\sqrt{211}}, y = -\frac{2}{\sqrt{211}}\right).$$

The values containing radicals in the denominator can be rationalized if desired, e.g.,

$$\frac{27}{\sqrt{211}} = \frac{27\sqrt{211}}{211}, \qquad \frac{2}{\sqrt{211}} = \frac{2\sqrt{211}}{211}.$$

Method 2. Let $y = tx$ in (1):

$$2x^2 + 5tx^2 - 10t^2x^2 = 8,$$
$$x^2 = \frac{8}{2 + 5t - 10t^2}. \tag{10}$$

Let $y = tx$ in (2):

$$x^2 - 2tx^2 + 3t^2x^2 = 3,$$
$$x^2 = \frac{3}{1 - 2t + 3t^2}. \tag{11}$$

Equate the right-hand members of (10) and (11):

$$\frac{8}{2 + 5t - 10t^2} = \frac{3}{1 - 2t + 3t^2},$$
$$8 - 16t + 24t^2 = 6 + 15t - 30t^2,$$
$$54t^2 - 31t + 2 = 0,$$
$$t = \frac{1}{2},\ \frac{2}{27}.$$

Substitute $t = \frac{1}{2}$ in (11). (We could use (10) just as well.)

$$x^2 = 4, \qquad x = \pm 2.$$
$$y = tx = \frac{1}{2}x = \pm 1.$$

Substitute $t = \frac{2}{27}$ in (11):

$$x^2 = \frac{729}{211}, \qquad x = \pm\frac{27}{\sqrt{211}}.$$
$$y = tx = \frac{2}{27}x = \pm\frac{2}{\sqrt{211}}.$$

The values must be paired as before.

The discerning student will note that methods 1 and 2 are essentially the same.

EXERCISES VII. C

Solve for x and y:

1. $x^2 + y^2 = 25,$
$xy = 12.$

2. $x^2 + 2xy = 5,$
$xy + 3y^2 = 2.$

3. $x^2 + xy = 32,$
$xy + y^2 = 32.$

4. $x^2 - xy - 98 = 0,$
$xy - y^2 + 98 = 0.$

5. $x^2 + xy - 3y^2 + 9 = 0,$
$x^2 - y^2 - 27 = 0.$

6. $x^2 + 3xy + 2y^2 = 3,$
$x^2 - 3xy - y^2 = 3$

7. $x^2 - xy + y^2 = 9,$
$3x^2 + 4xy = 33.$

8. $2x^2 - 9y^2 = 18,$
$3x^2 - 4xy = -3.$

9. $5xy - 3y^2 = -2,$
$2x^2 - 3y^2 = -10.$

10. $x^2 - xy - 4y^2 = 2,$
$3x^2 + 3xy + 2y^2 = 8.$

11. $7x^2 - 5xy + 9y^2 = 47,$
$4x^2 - 3xy + 5y^2 = 27.$

12. $3x^2 - 4xy + 3y^2 = 42,$
$x^2 - y^2 = -16.$

13. $7x^2 + 6xy + 6y^2 = 63,$
$4x^2 - 3xy + 2y^2 = 81.$

14. $4x^2 + 7xy + 5y^2 = 140,$
$x^2 + 3xy + 3y^2 = 52.$

15. The area of a rectangle is 168 square inches, its diagonal is 25 inches. What are its dimensions?

16. The area of a right triangle is 120 square inches. The median to one of its sides is 17 inches. Find its two sides.

62. Symmetric equations.

An equation is **symmetric** in two unknowns if the equation is unaltered when the two unknowns are interchanged;* for example,

$$3x^2 - 2xy + 3y^2 + x + y - 7 = 0.$$

Simultaneous quadratic equations both of which are symmetric can be solved by replacing one unknown by $u + v$, the other by $u - v$.

Example.

Solve:

$$3x^2 - 2xy + 3y^2 + x + y - 50 = 0, \quad (1)$$
$$2x^2 + 2y^2 - 3x - 3y - 29 = 0. \quad (2)$$

Solution. Let $x = u + v$, $y = u - v$.

From (1), $4u^2 + 8v^2 + 2u - 50 = 0.$ (3)

(3) ÷ 2 $2u^2 + 4v^2 + u - 25 = 0.$ (4)

From (2), $4u^2 + 4v^2 - 6u - 29 = 0.$ (5)

* Note that some symmetric simultaneous quadratics may also be otherwise classified. Thus, the equations

$$x^2 + y^2 = 25, \quad xy = 12$$

are of the type considered in the previous section and can be solved by the methods of that section or by the methods of the present section.

(5) − (4) $$2u^2 - 7u - 4 = 0. \tag{6}$$

$$u = 4, \ -\frac{1}{2}.$$

Substitute $u = 4$ in (4):

$$4v^2 = -2u^2 - u + 25 = -11,$$

$$v = \pm \frac{i}{2}\sqrt{11}.$$

$$x = u + v = 4 \pm \frac{i}{2}\sqrt{11},$$

$$y = u - v = 4 \mp \frac{i}{2}\sqrt{11}.$$

Substitute $u = -\frac{1}{2}$ in (4):

$$4v^2 = 25, \qquad v = \pm\frac{5}{2}.$$

$$x = -\frac{1}{2} + \frac{5}{2} = 2, \qquad y = -\frac{1}{2} - \frac{5}{2} = -3.$$

$$x = -\frac{1}{2} - \frac{5}{2} = -3, \qquad y = -\frac{1}{2} + \frac{5}{2} = 2.$$

The answers must be paired as follows:

$$\left(x = 4 + \frac{i}{2}\sqrt{11}, \quad y = 4 - \frac{i}{2}\sqrt{11}\right),$$

$$\left(x = 4 - \frac{i}{2}\sqrt{11}, \quad y = 4 + \frac{i}{2}\sqrt{11}\right),$$

$$(x = 2, y = -3), \quad (x = -3, y = 2).$$

It will be noted that the solutions are symmetric, as is necessarily the case.

It will be helpful to note that in symmetric quadratic equations the unknowns can always be grouped into the forms $x^2 + y^2$, xy, $x + y$, and that for the substitution $x = u + v$, $y = u - v$, we have

$$x^2 + y^2 = 2u^2 + 2v^2, \qquad xy = u^2 - v^2, \qquad x + y = 2u.$$

EXERCISES VII. D

Solve for x and y:

1. $x^2 - xy + y^2 = 19,$
   $xy + 6 = 0.$
2. $2x^2 + 2y^2 - 5x - 5y = 5,$
   $x^2 + y^2 + 2x + 2y = 7.$
3. $x^2 + y^2 - 3x - 3y = 14,$
   $xy + 4x + 4y = 56.$
4. $x^2 - 3xy + y^2 + x + y = 12,$
   $xy - 4x - 4y = -20.$
5. $3x^2 - xy + 3y^2 + 4x + 4y - 49 = 0,$
   $x^2 + 7xy + y^2 - 2x - 2y + 31 = 0.$
6. $4x^2 - 2xy + 4y^2 - 3x - 3y - 16 = 0,$
   $2x^2 - 4xy + 2y^2 + 3x + 3y - 2 = 0.$
7. $x^2 - 3xy + y^2 + x + y - 11 = 0,$
   $3x^2 + xy + 3y^2 - 2x - 2y - 13 = 0.$
8. $2x^2 + 3xy + 2y^2 - 5x - 5y + 1 = 0,$
   $x^2 - 2xy + y^2 + 4x + 4y + 20 = 0.$
9. $(x + y)^2 - 2(x - y)(y - x) - 4(x + y) - 71 = 0,$
   $x(y - x) + y(x - y) + 3(x + y) + 4 = 0.$
10. $3(x + y)^2 + 2(x - y)^2 - 4(x + y) - 260 = 0,$
    $2(x - y)^2 + 3(x + y) - 4xy + 70 = 0.$
11. The area of a right triangle is 210 square inches, its hypotenuse is 29 inches. Find its two sides.

63. Miscellaneous methods and types.

Sometimes special methods are shorter than those given above.

Example 1.

Solve:

$$x^2 + y^2 = 25, \tag{1}$$
$$xy = 12. \tag{2}$$

SOLUTION. These equations are symmetric, and they also come under the case in which all terms involving the unknowns are of second degree. They can therefore be solved as either of these cases. However, the following method is perhaps neater:

From (2), $$2xy = 24. \tag{3}$$

(1) + (3) $$x^2 + 2xy + y^2 = 49, \qquad x + y = \pm 7. \tag{4}$$

(1) − (3) $$x^2 - 2xy + y^2 = 1, \qquad x - y = \pm 1. \tag{5}$$

$$\begin{array}{llll} x + y = 7, & x + y = 7, & x + y = -7, & x + y = -7, \\ x - y = 1; & x - y = -1; & x - y = 1; & x - y = -1; \end{array}$$

$(x=4,\ y=3)$, $(x=3,\ y=4)$, $(x=-3,\ y=-4)$, $(x=-4,\ y=-3)$.

Example 2.

Solve:

$$x^3 + y^3 = 35,$$
$$x + y = 5.$$

Note that one equation is a cubic, the other linear.

SOLUTION. Divide the first equation by the second:

$$x^2 - xy + y^2 = 7.$$

Substitute $y = 5 - x$ in this last equation and reduce the result:

$$x^2 - 5x + 6 = 0,$$
$$x = 2, 3.$$
$$y = 3, 2.$$
$$(x = 2, y = 3), \qquad (x = 3, y = 2).$$

Example 3.

Solve:

$$6x^2 - 7xy + 2y^2 = 0, \tag{6}$$
$$y = x^2 - 4. \tag{7}$$

SOLUTION. In equation (6) each term is of degree two. Such an equation is called **homogeneous of degree two**. (Note that in homogeneous equations there can be no constant term, for such a term is of degree zero.) We can solve by setting $y = tx$ in the homogeneous equation, getting

$$6x^2 - 7tx^2 + 2t^2x^2 = 0. \tag{8}$$

Divide * (8) by x^2, getting

$$2t^2 - 7t + 6 = 0,$$
$$t = 2, \frac{3}{2}.$$

Substitute $y = 2x$ in (7):

$$x^2 - 2x - 4 = 0,$$
$$x = 1 \pm \sqrt{5}.$$
$$y = 2x = 2(1 \pm \sqrt{5}).$$

Substitute $y = \frac{3}{2}x$ in (7) and reduce:

$$2x^2 - 3x - 8 = 0,$$
$$x = \frac{3 \pm \sqrt{73}}{4}.$$
$$y = \frac{3}{2}x = \frac{9 \pm 3\sqrt{73}}{8}.$$

The complete solution is

$$[x = 1 + \sqrt{5}, y = 2(1 + \sqrt{5})], \quad [x = 1 - \sqrt{5}, y = 2(1 - \sqrt{5})],$$
$$\left(x = \frac{3 + \sqrt{73}}{4}, y = \frac{9 + 3\sqrt{73}}{8}\right),$$
$$\left(x = \frac{3 - \sqrt{73}}{4}, y = \frac{9 - 3\sqrt{73}}{8}\right).$$

* When we divide an equation by an expression involving an unknown we must set the expression equal to zero if possible. Here, therefore, we should set $x^2 = 0$. But $x = 0$ in (6) yields $y = 0$. The pair of values $(x = 0, y = 0)$ however does not satisfy (7).

EXERCISES VII. E

Solve for x and y:

1. $x^2 + y^2 = 74,$
$xy = 35.$

2. $x^2 + 4y^2 = 25,$
$xy = 6.$

3. $x^2 + xy = 50,$
$xy + y^2 = 50.$

4. $x^2 - xy = 63,$
$xy - y^2 = -18.$

5. $x^2 + 3xy = -27,$
$xy + 4u^2 = 52.$

6. $x^2 - 9y^2 = -35,$
$x + 3y = 5.$

7. $x^3 - y^3 = 98,$
$x - y = 2.$

8. $x^3 + 8y^3 = 280,$
$x + 2y = 10.$

9. $x^4 - y^4 = 80,$
$x^2 - y^2 = 8.$

10. $x^3 + y^3 = 218,$
$x^2 - xy + y^2 = 109.$

11. $x^2y^2 + 5xy = 6,$
$10x - 3y = 29.$

12. $x^3 + y^3 = 91,$
$x^2y + xy^2 = 84.$

13. $6x^2 - xy - 2y^2 = 0,$
$y = 2x^2 - 4.$

14. $10x^2 - 17xy + 3y^2 = 0,$
$y^2 = 3x - 5.$

15. $3x^2 - 4xy - 2y^2 = 18,$
$3x^2 - xy - 4y^2 = 0.$

16. $4x^2 + 4xy - 3y^2 = 0,$
$5x^2 + 3xy - 4y^2 = 33.$

17. $\dfrac{x}{x+y} - \dfrac{6y}{x-y} = 4,$
$(x + y)^2 = 4.$

18. $\dfrac{2x}{3x-4y} + \dfrac{5y}{3x+4y} = \dfrac{5}{2},$
$(x + y)(x - y) = 12.$

19. Find the positive square root of $5 + 2\sqrt{6}$.

SOLUTION. Let $\sqrt{x} + \sqrt{y} = \sqrt{5 + 2\sqrt{6}}$.

Square: $x + 2\sqrt{xy} + y = 5 + 2\sqrt{6}$.

It follows that

$$x + y = 5,$$
$$2\sqrt{xy} = 2\sqrt{6}.$$

These equations have the solutions $x = 3$, $y = 2$, and $x = 2$, $y = 3$. Thus, $\sqrt{5 + 2\sqrt{6}} = \sqrt{3} + \sqrt{2}$. This may be checked by squaring $\sqrt{3} + \sqrt{2}$.

Find the positive square roots of the following quantities:

20. $8 + 2\sqrt{15}$.

21. $11 + 6\sqrt{2}$.

22. $31 - 10\sqrt{6}$.

23. $9 - 2\sqrt{14}$.

24. $17 - 12\sqrt{2}$. **25.** $21 + 6\sqrt{10}$.
26. $30 + 12\sqrt{6}$. **27.** $385 + 150\sqrt{6}$.

64. Graphic representation.

Quadratic equations as well as linear equations can be represented graphically, although the plotting is usually more difficult. Graphic methods are of little use in the actual solution of equations but often give a clearer meaning to results. Points of intersection of curves correspond to pairs of *real* values of *both* variables satisfying the equations; *imaginary values do not appear on the graph.*

Points of tangency of the curves correspond to double solutions.

Example 1.

Draw the curves represented by, and solve, the following equations:

$$x^2 + y^2 = 25, \quad (1)$$
$$y = x^2 - 5. \quad (2)$$

Solution. The first equation is that of a circle with its center at the origin and having a radius of 5 units. For the coordinates x and y of any point on such a circle are (since $x^2 + y^2 = 5^2$) the sides of a right triangle whose hypotenuse is 5. (See Fig. 12.)

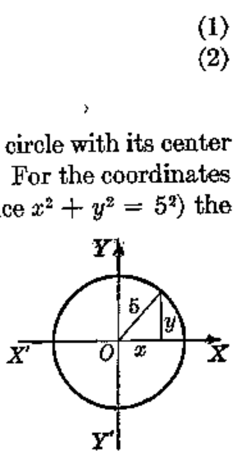


Fig. 12

$y = x^2 - 5$

x	y
0	−5
±1	−4
±2	−1
±3	4
±4	11

The other equation can be plotted by giving values to x and finding the corresponding values of y. A number of such values are shown in the accompanying table. The curve obtained by plotting them is a parabola. (See section 56.)

Both curves are shown in Fig. 13.

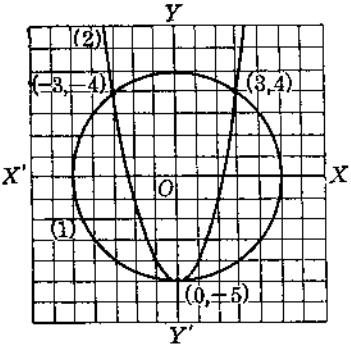


FIG. 13

Following is the algebraic solution:

From (2), $x^2 = y + 5.$ (3)

Substitute (3) in (1) and combine terms:

$$y^2 + y - 20 = 0,$$
$$y = -5, 4.$$

Substitute $y = -5$ in (3):

$$x^2 = 0, \qquad x = 0 \qquad \text{(double root)}.$$

Substitute $y = 4$ in (3):

$$x^2 = 9, \qquad x = \pm 3.$$

The values are paired as follows:

$$(x = 0, y = -5) \text{ (double solution)},$$
$$(x = 3, y = 4), (x = -3, y = 4).$$

Note that at the point corresponding to the double solution, the curves are tangent.

Example 2.

Plot and solve:

$$x^2 + y^2 = 25, \qquad (4)$$
$$4y^2 = 9x. \qquad (5)$$

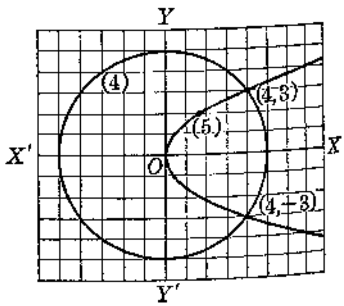


FIG. 14

SOLUTION. Equation (4) is the circle of example 1. Equation (5) is that of another parabola. Both curves are shown in Fig. 14. Following is the algebraic solution:

From (5), $$y^2 = \frac{9}{4}x.$$ (6)

Substitute in (4), clear of fractions, and transpose the constant term:

$$4x^2 + 9x - 100 = 0,$$

$$x = 4, -\frac{25}{4}.$$

Substitute $x = 4$ in (6):

$$y^2 = 9, \qquad y = \pm 3.$$

Substitute $x = -\frac{25}{4}$ in (6):

$$y^2 = -\frac{9}{4}\cdot\frac{25}{4}, \qquad y = \pm\frac{3}{2}\cdot\frac{5}{2}i = \pm\frac{15}{4}i.$$

The values are paired as follows:

$$(x = 4, y = 3), \quad (x = 4, y = -3), \quad \left(x = -\frac{25}{4}, y = \frac{15}{4}i\right),$$
$$\left(x = -\frac{25}{4}, y = -\frac{15}{4}i\right).$$

All four pairs of values satisfy both equations, but the last two pairs, in which y is imaginary, do not appear on the graph.

The subject of graphic representation of equations belongs more properly to the field of analytic geometry, in which simpler and more systematic methods than point-by-point plotting are developed. Consequently we shall not go further into the discussion here.

EXERCISES VII. F

The instructor may require the student to draw graphs for certain of the preceding exercises of the chapter.

For what values of k will the line and the curve, or the two curves, be tangent?

1. $x^2 + y^2 = 20,$ (7)
$y = 2x + k.$ (8)

SOLUTION. Equation (7) is represented by a circle with center at the origin and with radius $\sqrt{20} = 4.47$. Equation (8) is a series of parallel lines, one line corresponding to each value of k.

Substitute the value of y from (8) in (7):

$$x^2 + 4x^2 + 4kx + k^2 = 20,$$
$$5x^2 + 4kx + k^2 - 20 = 0. \qquad (9)$$

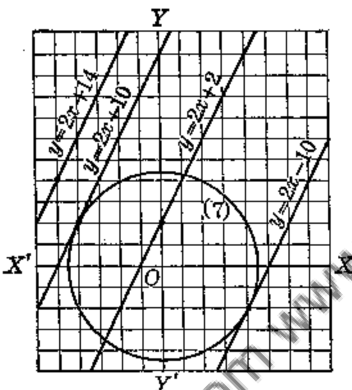


FIG. 15

Ordinarily there would be two distinct roots of this equation, x_1 and x_2, which, if real, would be the abscissas of the points in which the line cuts the circle. If the line is tangent to the circle these values of x would be the same, that is, equation (9) would have equal roots. Making use of the fact that if a quadratic equation has equal roots, its discriminant, $b^2 - 4ac$, must be zero, we find

$$16k^2 - 4 \cdot 5(k^2 - 20) = 0,$$

or

$$k^2 = 100, \qquad k = \pm 10.$$

The lines $y = 2x \pm 10$ will therefore be tangent to the circle. (See Fig. 15.)

2. $x^2 + y^2 = 40,$ $y = 3x + k.$

3. $x^2 + y^2 = 20,$ $y = kx + 10.$

4. $x^2 + y^2 = 25,$ $4x - 3y = k$

5. $x^2 + y^2 = k,$ $3x - 2y = 13.$

6. $y^2 = 8x,$ $y = 2x + k.$

7. $y^2 + 10x = 0,$ $y = 3x + k.$

8. $y^2 = 12x,$ $y = kx + 2.$

9. $y = 2x^2,$ $y = kx - 2.$

10. $y = -3x^2,$ $y = 2x + k.$

11. $y = 2x^2 - 3,$ $y = 4x + k.$

12. $9x^2 + 25y^2 = 225,$
$4x - 5y = k.$

13. $9x^2 - 16y^2 = 144,$
$5x - 4y = k.$

14. $y^2 = 2px,$
$y = mx + k.$

15. $x^2 + y^2 = r^2,$
$y = mx + k.$

16. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$
$y = mx + k.$

17. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$
$y = mx + k.$

MISCELLANEOUS EXERCISES VII. G

Solve:

1. $9x^2 - 6xy + 4y^2 = 63,$
$3x - 2y = 3.$

2. $x^2 + y^2 = 25,$
$3x + 4y = -15.$

3. $x^2 + y^2 = 40,$
$y = 2x^2 - 2.$

4. $4x^2 - 3y^2 = 24,$
$3x = 2y^2 + 1.$

5. $x^2 + y^2 - 2x - 6y - 15 = 0,$
$x - y + 3 = 0.$

6. $3x^2 + 4y^2 = 59,$
$5x^2 - 7y^2 = -11.$

7. $2x^2 + 3y^2 = 30,$
$3x^2 - 2xy = 39.$

8. $4x^2 - xy - 2y^2 = -6,$
$2xy + 3y^2 = 24.$

9. $2x^2 + 3xy + 2y^2 + 4x - 4y - 4 = 0,$
$3x^2 + 2xy + 3y^2 + 5x + 5y - 32 = 0.$

10. $2x^2 + xy + 9y^2 = 30,$
$x^2 + 3xy - 8y^2 = -20.$

11. $5x^2 + 9xy + 5y^2 + 28 = 0,$
$2x^2 - 3xy + 2y^2 + 64 = 0.$

12. $2x^2 - 7xy + 2y^2 - 3x - 3y + 24 = 0,$
$5x^2 + 2xy + 5y^2 - x - y - 44 = 0.$

13. $x^2 + y^2 = 148,$
$xy = 34.$

14. $9x^2 + 4y^2 + 277 = 0,$
$xy + 21 = 0.$

15. $8x^3 - y^3 = 387,$
$2x - y = 3.$

16. $x^3 + y^3 = 10xy,$
$x + y = xy.$

17. $2x^2y^2 - 5xy - 3 = 0,$
$x - 32y + 10 = 0.$

18. $x^3 - y^3 = 4xy,$
$x - y = xy.$

19. $\frac{1}{x^2} - \frac{1}{xy} + \frac{1}{y^2} = 3,$
$\frac{1}{x^2} + \frac{2}{xy} = 5.$

20. $x + \frac{1}{y} = 2,$
$y + \frac{1}{x} = 3.$

21. $6x^2 - 11xy - 10y^2 = 0,$
$2x^2 - 3y^2 - 4x = 18.$

22. $x^4 - y^4 = 0,$
$x^2 + y^2 = 72.$

23. $x^3 - y^3 = 98,$
$x^2y - xy^2 = 30.$

24. $x^3 + y^3 = 91,$
$x^2y + xy^2 = 84.$

25. $x^4 + y^4 = 706,$ (a)
$x + y = 2.$ (b)

26. $x^4 + y^4 = 337,$ (a)
$x + y = 7.$ (b)

SUGGESTION FOR SOLVING EXERCISES 25 AND 26. Raise (b) to fourth power and subtract (a). Divide by 2 and label resulting equation (c). Square (b), multiply by $2xy$, and subtract (c). Solve resulting equation for xy and combine with (b).

27. $5x^2 - 3y^2 + 2z^2 = -7,$
$4x^2 - 7y^2 + 3z^2 = -32,$
$3x^2 + 4y^2 - z^2 = 33.$

28. $x^2 + 2y^2 + 3z^2 = 20,$
$x + 2y + 3z = 4,$
$x + y + z = 4.$

29. $xy + x + y + 4 = 0,$
$yz + y + z + 3 = 0,$
$xz + x + z - 5 = 0.$

30. $xy + 2x + 3y = 10,$
$2yz + 3y + 4z = 30,$
$3xz + 4x + 5z = 28.$

31. $x^2 + y^2 + z^2 = 84,$
$x + y + z = 6,$
$y^2 = xz.$

32. $x^2 + xy + xz = 25,$
$y^2 + yz + xy = 40,$
$z^2 + xz + yz = 16.$

SUGGESTION. Add the three equations.

33. The area of a right triangle is 84 square inches, its hypotenuse is 25 inches. Find its two sides.

34. The product of two positive numbers is 144, the sum of their squares is 612. What are the numbers?

35. The product of two numbers is 288, the sum of their reciprocals is $\frac{1}{8}$. What are the numbers?

36. The perimeter of a rectangle is 41 inches, its area is 102 square inches. Find its dimensions.

37. The perimeter of a right triangle is 40 inches, its area is 60 square inches. Find its three sides.

38. A group of girls bought a \$30 wedding present for a friend. Two members of the group failed to pay their share and as a consequence each of the others had to pay \$1.25 more. How many girls were in the group originally?

39. A speculator sold some shares of stock for $9072. Several days later, the price of the stock having fallen $3 per share, he repurchased, for the same amount of money, 4 more shares than he had sold. How many shares did he sell?

40. If 3 is added to the numerator and to the denominator of a fraction and the result added to the original fraction the sum is $1\frac{1}{4}$. If 3 is subtracted from the numerator and from the denominator of a fraction and the result subtracted from the original fraction the difference is $\frac{5}{36}$. What is the fraction?

41. Two pipes together can fill a tank in 2 hours. The smaller pipe alone requires 1 hour and 10 minutes longer than the larger to fill it. How long does it take each pipe alone to fill it?

42. One day two clerks, A and B, were addressing envelopes. In the morning A had addressed 150 before B started in to help him. During the remainder of the morning they addressed 240 more, A working 3 hours altogether. In the afternoon A worked 3 hours and B 1 hour, in which time they addressed 360 envelopes. Find the average number of envelopes that each addressed per hour and the number of hours that B worked in the morning.

43. In a two-place number the square of the digit in tens' place is 5 less than the sum of the digits. If the digits are reversed the number is increased by 45. What is the number?

44. The sides of a triangle are 10, 12, 14 respectively. Find the medians.

45. The medians of a triangle are 4, 5, 6 respectively. Find the sides.

46. A sum of money invested for a year brought $54 interest. If the rate had been $\frac{1}{2}$ per cent more and the principal $360 less, the interest would have been the same. Find the principal and the rate.

47. A rod rests upon a fulcrum which is $1\frac{1}{2}$ feet from one end of the rod. A weight of 12 pounds suspended from this end of the rod causes the rod to balance. If the weight is suspended from the other end of the rod it is necessary to suspend a 56-pound weight from the first end in order to make the rod balance. Find the length and the weight of the rod. (See p. 26.)

48. The laborers in a certain trade have been receiving \$77 a week. They are striking for the same amount per week for 4 less hours per week, which would increase their hourly wage by $17\frac{1}{2}$ cents. What is their present hourly wage?

49. The area of the page of a book is $46\frac{3}{4}$ square inches, the area of the printed part is 27 square inches. The margin at the top and sides is $\frac{3}{4}$ inch wide, the margin at the bottom is 1 inch wide. Find the dimensions of the page.

50. In 5 hours a motor boat can either go 24 miles downstream and back or 27 miles downstream and 22 miles back. Find its rate in still water and the rate of the current.

51. A boat made a trip of 6 miles with the tide and back against the tide in 1 hour and 52 minutes. If the tide had been half as strong the trip would have been made in 7 minutes less time. Find the rate of the boat in still water and the rate of the stronger tide.

52. An airplane flew a certain distance at a uniform rate. If this rate had been 40 miles per hour faster the trip would have been made in 24 minutes less time. If the rate had been 20 miles per hour slower the trip would have required 16 minutes longer. Find the rate of the plane and the distance it flew.

53. Two airplanes started at the same time from airports A and B, the first going from A to B, the second going from B to A. Each traveled at a uniform rate. The first reached B 45 minutes after they passed each other, the second reached A 1 hour and 20 minutes after they passed each other. Find the time that each required to make the trip.

54. A man is at a point A on the bank of a straight river. He can row in still water at a rate which is 25 per cent faster than he can walk. If he rows downstream to a point B, which is 4 miles below A and on the same side of the river, and then walks, in a direction perpendicular to the bank of the river, to a point C, 3 miles from B, it takes him the same amount of time that it does to walk straight from A to C. If he walks back from C to B and then rows upstream to A, the return trip takes him an hour and a half longer than the trip from A to C. Find his rate of walking, his rate of rowing in still water, and the rate of the current.

CHAPTER VIII

Inequalities

65. Inequalities.

The statement "a is greater than b" ($a > b$), a and b being real numbers, means that $a - b$ is a positive number. If we have an axis (Fig. 16) on which the positive direction

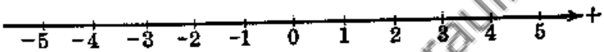


FIG. 16

is to the right, a will be to the right of b. Similarly, "a is less than b" ($a < b$) means that $a - b$ is a negative number, and on an axis directed toward the right, a will be to the left of b. Such expressions are called **inequalities.** *All inequalities refer to real numbers.*

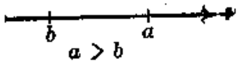


FIG. 17

Two inequalities $a > b$, $c > d$ in which the signs point in the same direction are said to be **alike in sense** (or **order**); two inequalities $a > b$, $c < d$ in which the signs point in opposite directions are said to be **opposite in sense** (or **order**).

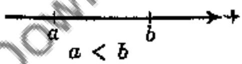


FIG. 18

The statement $a \geqq b$ is read "a is greater than or equal to b"; the statement $a \leqq b$ is read "a is less than or equal to b."

Some inequalities are satisfied by all real numbers, for example, $x^2 + 3 > 0$. Such inequalities are called **absolute inequalities.** Other inequalities, called **conditional inequalities,** are satisfied only by certain numbers. Thus, the inequality $x - 3 > 0$ is satisfied only by numbers greater than 3.

66. Fundamental properties.

The following properties are useful in dealing with inequalities. Their proofs are simple, following almost immediately from the definition of an inequality. The proof of Property I only will be given. Proofs of the others can be effected in a similar manner.

I. An inequality is unchanged in sense if the same number is added to or subtracted from both sides. That is, *if* $a > b$, *then* $a + c > b + c$ *and* $a - c > b - c$.

Proof. By hypothesis, $a > b$.
Let $a - b = p$, which will be positive, since $a > b$. Then

$$(a + c) - (b + c) = a - b = p.$$

That is,
$$a + c > b + c.$$

Similarly we can prove that $a - c > b - c$.

Example 1.

$$10 > 8,$$
$$10 + 3 > 8 + 3.$$

Example 2.

$$10 > 8,$$
$$10 - 3 > 8 - 3.$$

II. An inequality is unchanged in sense if both sides are multiplied or both divided by the same positive number.

Example 3.

$$2 < 3,$$
$$5 \cdot 2 < 5 \cdot 3.$$

Example 4.

$$3x \geqq 15,$$
$$x \geqq 5.$$

III. An inequality is changed in sense if both sides are multiplied or both divided by the same negative number.

Example 5.

$$2 < 3,$$
$$-2 > -3.$$

(Each side has been multiplied by -1.)

Example 6.

$$8 > 6,$$
$$8 \div (-2) < 6 \div (-2),$$
$$-4 < -3.$$

IV. An inequality of positive quantities is unchanged in sense if the same positive power or same positive root of each side is taken.

Example 7.

$$3 > 2,$$
$$3^2 > 2^2.$$

Example 8.

$$49 > 25,$$
$$\sqrt{49} > \sqrt{25}.$$

EXERCISES VIII. A

Prove the following:

1. *If the first of three quantities is greater than the second, and the second greater than the third, then the first is greater than the third;* that is, *if $a > b$ and $b > c$, then $a > c$.*
2. *If unequals are added to unequals in the same order, the sums are unequal in the same order;* that is, *if $a > b$ and $c > d$, then $a + c > b + d$.*
3. *If unequals are subtracted from equals, the results are unequal in the opposite order;* that is, *if $a > b$, then $c - a < c - b$.* (Show, by means of an example, that if $a > b$ and $c > d$, then it does not necessarily follow that $a - c > b - d$.)
4. *If $a > b \geqq 0$ and $c > d \geqq 0$, then $ac > bd$.*

67. Solution of inequalities.

Inequalities can be solved, by applying these fundamental properties, very much as equations are solved by the use of axioms concerning equals. Graphs are often of assistance, especially in the case of quadratic inequalities, in determining when an inequality is satisfied.

Example 1.

Find the values of x for which

$$3x - 4 > \frac{7x}{2} + 2.$$

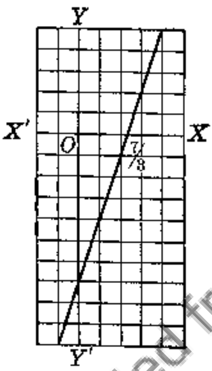


FIG. 19

SOLUTION.

Multiply by 2: $6x - 8 > 7x + 4.$

Subtract $7x$ and add 8: $-x > 12.$

Multiply by -1: $x < -12.$

Example 2.

Give a graphic interpretation of the inequality $3x - 7 > 0$.

SOLUTION. Set $y = 3x - 7$ and draw the graph. It is seen that the graph crosses the x-axis between 2 and 3. Solving algebraically, we find

$$3x > 7, \qquad x > \frac{7}{3}.$$

Example 3.

Find the values of x for which

$$x^2 + 3x - 4 > 0.$$

SOLUTION. Draw the graph (Fig. 20) of the function

$$y = x^2 + 3x - 4.$$

It is seen that the graph crosses the x-axis at -4 and 1 and that the function is positive (i.e., > 0) for values of x outside these values, namely,

$$x < -4 \text{ or } x > 1.$$

Algebraically we can solve by completing the square:

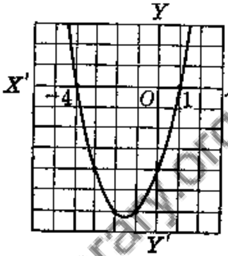


FIG. 20

$$x^2 + 3x > 4,$$

$$x^2 + 3x + \frac{9}{4} > 4 + \frac{9}{4},$$

$$\left(x + \frac{3}{2}\right)^2 > \frac{25}{4}.$$

This last inequality will hold if

$$x + \frac{3}{2} > \frac{5}{2} \quad \text{or if} \quad x + \frac{3}{2} < -\frac{5}{2};$$

that is, if

$$x > 1 \quad \text{or if} \quad x < -4.$$

A satisfactory way of solving a quadratic inequality is to replace the inequality sign by an equality sign, solve the resulting equation, and note from the graph, or by testing with numbers, whether the inequality is satisfied between the roots or outside of the roots.

68. The general quadratic function.

Consider the general quadratic function,

$$ax^2 + bx + c, \tag{1}$$

which may also be written in the form

$$a(x - x_1)(x - x_2), \tag{2}$$

x_1 and x_2 being the roots of the equation obtained by setting the function equal to zero.

If these roots are imaginary, the function cannot change sign. For we recall (see section 56) that when a quadratic equation has imaginary roots, the curve representing the corresponding quadratic function is wholly above or wholly below the x-axis (because if it crossed, the equation would have a real root). It must therefore have the sign of c, since when $x = 0$ the function is equal to c. But if the roots are imaginary we must have $b^2 - 4ac < 0$. This requires that a and c have the same sign, since they are real numbers. Therefore we can state that in this case $ax^2 + bx + c$ has the same sign as a and c for all values of x.

If the roots are equal ($x_1 = x_2$), we may write the function in the form

$$a(x - x_1)^2. \tag{3}$$

This will be positive when and only when a is positive and $x \neq x_1$. (The symbol $\neq$ means "is not equal to.")

If the roots are real and different and a is positive, suppose $x_1 < x_2$. From (2) we see that the factors $x - x_1$ and $x - x_2$ will both be negative when $x < x_1$, and that consequently for such values of x the function will be positive. When x is between x_1 and x_2, $x - x_1$ will be positive and $x - x_2$ will be negative. The expression (2) will then be the product of three factors having the signs $+$, $-$, $+$, respectively, and the function will be negative. When $x > x_2$ all the factors will be positive and the function will be positive.

If a is negative the situation is reversed, and the function $a(x - x_1)(x - x_2)$ will be positive only when x is between x_1 and x_2. Results are summarized in the table below:

If $b^2 - 4ac$ is	Then $ax^2 + bx + c$ will have same sign as a for
$-$	All real values of x.
0	$x \neq x_1$.
$+$	$x < x_1$, $x > x_2$ (if $x_1 < x_2$).

EXERCISES VIII. B

Solve the following inequalities:

1. $3x > 7$.
2. $2x - 1 > 5$.
3. $7x + 3 > 5x + 6$.
4. $3x + 1 > 5x - 4$.
5. $\dfrac{3x+4}{3} < \dfrac{5x-6}{2}$.
6. $\dfrac{2-3x}{5} > \dfrac{6-x}{7}$.
7. $x^2 - 5x - 6 > 0$.
8. $x^2 - 6x + 8 < 0$.
9. $x^2 - 4x + 2 > 0$.
10. $12 - x - x^2 > 0$.
11. $x^2 - 2x - 5 > 10$.
12. $6x - x^2 < 9$.
13. $2x^2 + 5x - 12 > 0$.
14. $12x^2 + 11x - 12 < 3$.
15. $2x^2 - 6x + 7 > 0$.
16. $3x^2 - 4x + 5 < 0$.
17. $10(x^2 - x + 1) > 2x^2 + 20x - 17$.
18. $5x^2 + 6x + 7 < 3x^2 + 2x + 10$.
19. $(x + 2)(x - 1)(x - 3) > 0$.
20. $(x - 1)^2(x - 3) > 0$.
21. $(x - 1)(x - 3)^2 > 0$.
22. $(x - 1)^3(x - 3)^2 > 0$.
23. $x(x^2 - 1) > 0$.
24. $(x + 3)^2(x - 1)(x - 4) > 0$.
25. $(x + 3)(x - 1)^2(x - 4) > 0$.
26. $(x + 3)(x - 1)(x - 4)^2 > 0$.
27. $(x + 3)(x - 1)^2(x - 4)^2 > 0$.
28. $\dfrac{3}{x+2} > \dfrac{2}{x-1}$.

SOLUTION. Subtract $2/(x - 1)$ from both sides:

$$\frac{3}{x+2} - \frac{2}{x-1} > 0.$$

Clear of fractions by multiplying both sides by the expression $(x + 2)^2 (x - 1)^2$, which is positive unless x is equal to 1 or -2:

$$\begin{aligned} 3(x + 2)(x - 1)^2 - 2(x + 2)^2(x - 1) &> 0, \\ (x + 2)(x - 1)[3(x - 1) - 2(x + 2)] &> 0, \\ (x + 2)(x - 1)(x - 7) &> 0. \end{aligned}$$

By studying the signs of the three factors when x is in various intervals, we see that the inequality is satisfied for

$$-2 < x < 1 \qquad \text{and} \qquad x > 7.$$

29. $\dfrac{3}{x-2} > \dfrac{2}{x+3}$.

30. $\dfrac{5}{2x+3} > \dfrac{3}{4x-7}$.

31. $\dfrac{1}{x^2} + \dfrac{2}{x-3} > 0$.

32. $\dfrac{3}{x+4} > \dfrac{2}{x^2}$.

33. $\dfrac{x+1}{x-3} > \dfrac{x-5}{x+7}$.

34. $\dfrac{x-1}{x-3} > \dfrac{x-3}{x-5}$.

Find the values of k for which the roots of the following quadratic equations are real and different:

SUGGESTION. The discriminant must be positive.

35. $x^2 + kx = k$.

36. $16x^2 - 6x + k = 0$.

37. $x^2 + k^2 + x + k = 0$.

38. $(k+1)x^2 + (x+1)k^2 = 0$.

39. $x^2 + k(x+1) + x(k+1) + k = 0$.

40. $kx^2 + kx - 5x^2 + k - 5 = 0$.

Find the values of k for which the roots of the following quadratic equations are imaginary:

41. $kx^2 + 2x - 3 = 0$.

42. $25x^2 + 5x + 1 = kx$.

43. $x^2 + k^2 + kx + 3x = 3k$.

44. $(k-3)x^2 - (x-3)k^2 = 0$.

Find, in the following exercises, the values of k for which the line intersects the curve in two distinct points:

SUGGESTION. The roots of the quadratic equation obtained by substituting from the linear equation in the quadratic must be real and different.

45. $x^2 + y^2 = 40$,
$y = 3x + k$.

46. $x^2 + y^2 = 36$,
$y = 2x + k$.

47. $y^2 = 8x$,
$y = 2x + k$.

48. $x^2 = 8y$,
$y = 2x + k$.

49. $x^2 + y^2 = 25$,
$4x - 3y = k$.

50. $y^2 + 12x = 0$,
$y = kx + 2$.

51. $x^2 + y^2 = 49$,
$y = kx + 25$.

52. $x^2 - y^2 = 36$,
$y = kx + 8$.

53. $x^2 + 24y^2 = 24$,
$y = x + k$.

54. $16x^2 - 25y^2 = 400$,
$y = x + k$.

CHAPTER IX

Proportion and Variation

69. Ratio.

The **ratio** of a to b is the quotient $a \div b$, that is, the fraction a/b. The ratio may also be written $a:b$.

70. Proportion.

A **proportion** is a statement that two ratios are equal, for example,

$$\frac{a}{b} = \frac{c}{d}.$$

The quantities b and c are called the **means** of the proportion, a and d are called the **extremes.**

The quantity d is called the **fourth proportional** to a, b, and c.

If $\frac{a}{b} = \frac{b}{x}$, x is called the **third proportional** to a and b.

If $\frac{a}{x} = \frac{x}{b}$, x is called a **mean proportional** between a and b.

71. Theorems on proportion.

Given

$$\frac{a}{b} = \frac{c}{d}; \qquad (1)$$

prove the following theorems:

I. $ad = bc.$ i.e., *The product of the means is equal to the product of the extremes.*

II. $\dfrac{a}{c} = \dfrac{b}{d}.$ This proportion is said to be obtained from (1) by **alternation.**

III. $\dfrac{b}{a} = \dfrac{d}{c}.$ This is said to be obtained from (1) by **inversion.**

IV. $\dfrac{a+b}{b} = \dfrac{c+d}{d}.$ This is said to be obtained from (1) by **composition.** SUGGESTION. Add 1 to each side of (1).

V. $\dfrac{a-b}{b} = \dfrac{c-d}{d}.$ This is said to be obtained from (1) by **division.** SUGGESTION. Subtract 1 from each side of (1).

VI. $\dfrac{a+b}{a-b} = \dfrac{c+d}{c-d}.$ This is said to be obtained from (1) by **composition and division.**

Given
$$\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \frac{g}{h};$$
prove

VII.
$$\frac{a+c+e+g}{b+d+f+h} = \frac{a}{b}.$$

72. Direct variation.

Each of the statements "y varies directly as x," "y varies as x," "y is directly proportional to x," "y is proportional to x," in a mathematical sense means

$$y = kx, \qquad (1)$$

in which k is a constant called the **constant of variation** or **constant of proportionality.** (The statement "y varies as x" is sometimes written $y \propto x$, but the form (1) is decidedly preferable.) The constant k in (1) can be determined if any pair of corresponding values of x and y, except $x = 0$, $y = 0$, is known.

The statement "y varies as the square of x" means $y = kx^2$.

73. Inverse variation.

The statements "y varies **inversely** as x," and "y is inversely proportional to x" mean

$$y = \frac{k}{x}\cdot$$

74. Joint and combined variation.

The statement "z varies **jointly** as x and y" means

$$z = kxy.$$

The statement "w varies directly as x and the cube of y and inversely as the square root of z" means

$$w = \frac{kxy^3}{\sqrt{z}}\cdot$$

EXERCISES IX. A

1. If y varies as x and is equal to 11 when x is 7, find the value of y when x is equal to 5.

SOLUTION. $y = kx.$

$$11 = 7k, \quad k = \frac{11}{7}\cdot$$

Having determined the constant k, we can state the law of variation:

$$y = \frac{11}{7}x.$$

This law is now ready for use. For example, when $x = 5$ we have

$$y = \frac{11}{7}\cdot 5 = \frac{55}{7}\cdot$$

2. The pressure on a sail varies as the area of the sail and as the square of the velocity of the wind. When the velocity of the wind is 36 feet per second, the pressure on a square foot of sail is 3 pounds. Find (a) the pressure on a square yard when the velocity is 20 feet per second, (b) the velocity of the wind when the pressure is 10 pounds per square foot.

SOLUTION.

$$p = kav^2.$$

$$3 = k \cdot 1(36)^2, \qquad k = \frac{3}{(36)^2}.$$

$$p = \frac{3}{(36)^2} av^2.$$

(a) $$p = \frac{3}{(36)^2} 9(20)^2 = \frac{25}{3} = 8\frac{1}{3} \text{ lb.}$$

(b) $$10 = \frac{3}{(36)^2} 1 \cdot v^2,$$

$$v^2 = \frac{10(36)^2}{3},$$

$$v = 36\sqrt{\frac{10}{3}} = 12\sqrt{30} = 65.7 \text{ ft. per sec.}$$

3. If y varies as x and is equal to 20 when x is equal to 15, what is the value of y when x is equal to 12? What is the value of x when y is equal to 12?

4. If y varies as x and is equal to 3 when x is equal to 7, what is the value of y when x is equal to $4\frac{2}{3}$?

5. If y varies as the square of x and is equal to 12 when x is equal to 4, what is the value of y when x is equal to 2; when x is equal to 3?

6. The quantity z varies jointly as x and y and has the value 12 when x is equal to 3 and y is equal to 5. (a) Find the value of z when x is equal to 10 and y is equal to 6. (b) Find the value of y when x is equal to 21 and z is equal to 14.

7. If y varies as the square root of x and is equal to 16 when x is equal to 25, what is the value of y when x is equal to 9? What is the value of x when y is $\frac{4}{5}$?

8. If y varies as the square root of x and has the value $\sqrt{2}$ when x has the value 8, what is y equal to when x is equal to $\sqrt{2}$?
9. If y varies inversely as x and has the value 24 when x is equal to $\frac{2}{3}$, what is the value of y when x is equal to $2\frac{2}{3}$?
10. The quantity z varies directly as x and inversely as y; z is equal to 12 when x is equal to 4 and y is equal to 3. Find the value of z (a) when x is 3 and y is 4, (b) when x is 6 and y is 27.
11. The quantity z varies directly as the square root of x and inversely as the cube of y. If z is 4 when x and y have the values 3 and 2 respectively, what is the value of z when x is 27 and y is 4?
12. Hooke's law states that the extension of a spring or an elastic string beyond its natural length varies as the force applied. If a weight of 6 pounds attached to a spring causes it to stretch from a length of 10 inches to a length of 11.5 inches, what weight will cause it to stretch to a length of 1 foot?
13. The power required to drive a fan varies as the cube of the speed. If 1 horsepower will drive a fan at a speed of 470 revolutions per minute, how fast will 8 horsepower drive it?
14. If it requires 0.1 horsepower to drive an electric fan at a speed of 1600 revolutions per minute, what horsepower will be required to drive it at a speed of 1200 revolutions per minute? (See preceding exercise.)
15. The force of the wind blowing perpendicularly onto a flat surface varies directly as the area of the surface and the square of the velocity of the wind. When the wind is blowing 20 miles per hour the force on an area of 1 square yard is 18 pounds. What is the force on 1 square foot when the wind is blowing (a) 16 miles per hour? (b) 25 miles per hour? (c) What is the velocity of a wind which produces a pressure of 8 pounds per square foot?
16. The complete period of a simple pendulum (the time required for it to swing across and back) varies directly as the square root of its length. A pendulum 1 foot long has a period of 1.11 seconds. (a) What is the period of a pendulum whose length is 1 yard? (b) What is the length of a pendulum whose period is 2 seconds?
17. If it takes 8 days for a man to dig a hole 8 feet square and 8 feet deep, how long will it take him to dig a hole 4 feet square

and 4 feet deep? Assume that the time required is proportional to the volume of the hole.

18. It takes 11 weeks for 14 men, working 8 hours a day for 6 days a week, to complete a certain job. How many weeks will it take 24 men, working 7 hours a day for $5\frac{1}{2}$ days a week, to complete the job?

19. If one weight draws up another by means of a string passing over a fixed pulley, the distance passed over by each weight in a given time varies directly as the difference between the weights and inversely as their sum. If a 9-pound weight lifts a 7-pound weight through a distance of 8 feet in 2 seconds, how far will a 5-pound weight lift a 3-pound weight in the same time?

20. The power required to propel a ship is proportional to the cube of the speed. (a) By what per cent must the power be increased to increase the speed 10 per cent? (b) If a speed of 8 knots requires 1600 horsepower, what horsepower will be required for a speed of 10 knots?

21. Boyle's law states that at a constant temperature the pressure of a gas varies inversely as its volume. If a certain amount of gas is under a pressure of 20 pounds per square inch and has a volume of 225 cubic inches, what will be the pressure if the volume is (a) decreased to 200 cubic inches? (b) increased to 250 cubic inches?

22. (a) By what per cent must the volume of a gas be decreased if the pressure is to be increased 50 per cent? (b) By what per cent will the pressure of a gas be increased if the volume is decreased 20 per cent? (See preceding exercise.)

23. The safe load for a beam supported at both ends varies directly as the breadth and the square of the depth and inversely as the distance between supports. If a 2×6 (inches) wooden beam whose supports are 12 feet apart has a safe load of 900 pounds, what is the safe load for a 4×8 beam of the same material for which the distance between supports is 15 feet?

24. The quantity of water discharged over a weir, in a given interval of time, varies directly as the length of the weir and the three-halves power of the head. (The head is the difference in height between the level of the crest of water flowing over the weir and the level of still water above the weir.) If the rate of discharge of water over a weir 3 feet long and having a head of 9 inches is 390 cubic feet per minute, what is the rate of discharge of a weir 5 feet long and having a head of 1 foot 4 inches?

25. The weight of an object above the surface of the earth varies inversely as the square of its distance from the center of the earth. If it were possible to project an object which weighs 10 pounds at the surface of the earth to a point 1000 miles above the surface, what would it weigh at that height? (Assume the radius of the earth to be 4000 miles.)

26. The surfaces of similar solids vary as the squares of corresponding linear dimensions, their volumes vary as the cubes of corresponding linear dimensions. Derive a formula which describes the variation of surfaces of similar solids with respect to their volumes.

27. The horsepower that can be safely transmitted by a shaft varies directly as the number of revolutions it makes per minute and the cube of its diameter. If a shaft 3 inches in diameter making 200 revolutions per minute can safely transmit 60 horsepower, what horsepower can be safely transmitted by a 2-inch shaft making 300 revolutions per minute?

28. The electrical resistance of a wire varies directly as its length and inversely as the square of its diameter. If a copper wire 100 feet long and 0.024 inch in diameter has a resistance of 1.8 ohms, what is the resistance of a wire 150 feet long and 0.036 inch in diameter?

29. The amount of electrical current required to melt a fuse wire varies as the three-halves power of the diameter. If the current required to melt a wire of diameter 0.06 inch is 27 amperes, what current will melt a wire of diameter 0.04 inch?

30. The so-called standard error of the arithmetic mean of a sample varies inversely as the square root of the number in the sample. (a) If the standard error of the arithmetic mean of a sample of 20 items is 3.24, what is the standard error of the arithmetic mean of a sample of 45 items? (b) What sample size would be necessary to reduce the standard error to 1.08?

31. Kepler's third law states that the square of the time required for a planet to revolve around the sun varies as the cube of its distance from the sun. Jupiter is 5.2 times as far from the sun as the earth is. How long does it take Jupiter to revolve about the sun?

32. The number of vibrations made by a stretched string varies directly as the square root of the stretching force, or tension, and inversely as the product of the length and the diameter. (a) If a string 3 feet long and 0.05 inch in diameter vibrates 720 times per second under a tension of 90 pounds, how many vibrations per second will be made by a 2-foot string 0.03 inch in diameter when under a tension of 40 pounds? (b) To what tension must a string 1 foot 3 inches long and 0.09 inch in diameter be subjected to make it vibrate 640 times per second?

33. The illumination from a source of light varies inversely as the square of the distance from the source. A piece of cardboard is midway between two sources of light of equal intensity which are at a distance d apart. Its plane is perpendicular to the line joining the two sources. How far must it be moved, being kept always in this line and perpendicular to it, so that the total illumination which it receives (i.e. on both sides) will be ten times as great?

CHAPTER X

Mathematical Induction and the Binomial Formula

75. Mathematical induction.

An extremely important type of reasoning is the process called **mathematical induction.** It is extremely powerful in effecting certain proofs, especially in establishing the validity of certain types of formulas. We shall illustrate the method by proving that the sum of the first n odd integers is equal to n^2, that is,

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2. \qquad (1)$$

The proof consists of two parts:

Part I. The formula is true for certain particular values of n, e.g.,

$$\begin{array}{llll} \text{for } n = 1, & 1 = 1^2, & \text{or} & 1 = 1; \\ \text{for } n = 2, & 1 + 3 = 2^2, & \text{or} & 4 = 4; \\ \text{for } n = 3, & 1 + 3 + 5 = 3^2, & \text{or} & 9 = 9. \end{array}$$

Part II. Let us suppose that it is true for some integral value of n, say $n = k$; that is,

$$1 + 3 + 5 + \cdots + (2k - 1) = k^2. \qquad (2)$$

Then we can show that it is true for the next integral value of n, namely, $k + 1$.

If $n = k + 1$, then $2n - 1 = 2(k + 1) - 1 = 2k + 1$.

Add $2k + 1$ to both sides of (2):

$$1 + 3 + 5 + \cdots + (2k - 1) + (2k + 1) = k^2 + 2k + 1 = (k + 1)^2. \quad (3)$$

Thus, if formula (1) is true for $n = k$, it is true for $n = k + 1$. But we have seen that it is true for the particular values $n = 1, 2, 3$. Therefore, since it is true for $n = 3$, it must, by Part II, be true for $n = 4$; since true for $n = 4$, it must be true for $n = 5$, and so on for any positive integral value of n whatever.

In general, a proof by mathematical induction consists of two parts:

Part I. *A verification that the proposition or formula is true for the least integral value of n for which it is to hold.*

Part II. *A proof that if it is true for any integral value of n, say $n = k$, then it is true for the next value, $n = k + 1$.*

The formula will then be true for all values of n from the verified value on up.

The two parts of the proof do not have to be established in the order in which they have been given, as they are quite independent of each other. Both parts, however, are absolutely necessary to the proof, as is shown by the following illustrations:

Consider the formula

$$2 + 4 + 6 + \cdots + 2n = n^3 - 5n^2 + 12n - 6.$$

We have

for $n = 1$,

$$2 = 1^3 - 5 \cdot 1^2 + 12 \cdot 1 - 6, \qquad \text{or} \qquad 2 = 2;$$

for $n = 2$,

$$2 + 4 = 2^3 - 5 \cdot 2^2 + 12 \cdot 2 - 6, \qquad \text{or} \qquad 6 = 6;$$

for $n = 3$,

$$2 + 4 + 6 = 3^3 - 5 \cdot 3^2 + 12 \cdot 3 - 6, \qquad \text{or} \qquad 12 = 12.$$

Thus, we might be tempted to assume that the formula is true for all integral values of n. However, Part II of the mathematical induction proof is impossible to demonstrate; the formula is not true for any values of n other than 1, 2, 3. If we substitute $n = 4$, for example, we are led to the contradictory result $20 = 26$.

Similarly, the formula

$$2 + 4 + 6 + \cdots + 2n$$
$$= n(n + 1) + (n - 1)(n - 2) \cdots (n - 100)$$

holds for all positive integral values of n up to and including 100, but fails for $n = 101$.

On the other hand, consider the formula

$$1 + 2 + 3 + \cdots + n = \frac{1}{2}\, n(n + 1) + 1.$$

Assume that it is true for $n = k$. Then

$$1 + 2 + 3 + \cdots + k = \frac{1}{2}\, k(k + 1) + 1.$$

Add $k + 1$ to each side:

$$1 + 2 + 3 + \cdots + k + (k + 1)$$
$$= \frac{1}{2}\, k(k + 1) + 1 + (k + 1) = \frac{1}{2}\,(k + 1)(k + 2) + 1.$$

Thus, the formula is true for $n = k + 1$ if it is true for $n = k$, and Part II of the proof has been demonstrated. However, Part I cannot be proved, since there can be found no value of n for which the " formula " is true.

EXERCISES X. A

Prove by mathematical induction:

1. $1 + 2 + 3 + \cdots + n = \frac{1}{2}n(n + 1)$.

2. $4 + 8 + 12 + \cdots + 4n = 2n(n + 1)$.

3. $1 + 5 + 9 + \cdots + (4n - 3) = n(2n - 1)$.

4. $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{6}n(n + 1)(2n + 1)$.

5. $1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2$.

6. $2^0 + 2^1 + 2^2 + \cdots + 2^{n-1} = 2^n - 1$.

7. $2^{-1} + 2^{-2} + 2^{-3} + \cdots + 2^{-n} = 1 - 2^{-n}$.

8. $3 + 3^2 + 3^3 + \cdots + 3^n = \frac{3}{2}(3^n - 1)$.

9. $1 \cdot 2 + 3 \cdot 4 + 5 \cdot 6 + \cdots + (2n - 1)2n$
$= \frac{1}{3}n(n + 1)(4n - 1)$.

10. $2 \cdot 4 + 4 \cdot 6 + 6 \cdot 8 + \cdots + 2n(2n + 2)$
$= \frac{4}{3}n(n + 1)(n + 2)$.

11. $\dfrac{1}{2 \cdot 4} + \dfrac{1}{4 \cdot 6} + \dfrac{1}{6 \cdot 8} + \cdots + \dfrac{1}{2n(2n + 2)} = \dfrac{n}{4(n + 1)}$.

12. $1 \cdot 3 + 3 \cdot 5 + 5 \cdot 7 + \cdots + (2n - 1)(2n + 1)$
$= \frac{1}{3}n(4n^2 + 6n - 1)$.

13. $\dfrac{1}{1 \cdot 3} + \dfrac{1}{3 \cdot 5} + \dfrac{1}{5 \cdot 7} + \cdots + \dfrac{1}{(2n - 1)(2n + 1)} = \dfrac{n}{2n + 1}$.

14. $1 \cdot 3 + 2 \cdot 3^2 + 3 \cdot 3^3 + \cdots + n \cdot 3^n = \frac{3}{4}[(2n - 1) \cdot 3^n + 1]$.

15. $2 \cdot 1^2 + 3 \cdot 2^2 + 4 \cdot 3^2 + \cdots + (n + 1)n^2$
$= \frac{1}{12}n(n + 1)(n + 2)(3n + 1)$.

16. $1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6 + \cdots + n(n + 3)$
$= \frac{1}{3}n(n + 1)(n + 5)$.

17. $\dfrac{1}{1 \cdot 2 \cdot 3} + \dfrac{1}{2 \cdot 3 \cdot 4} + \dfrac{1}{3 \cdot 4 \cdot 5} + \cdots + \dfrac{1}{n(n + 1)(n + 2)}$
$= \dfrac{n(n + 3)}{4(n + 1)(n + 2)}$.

18. $\dfrac{4}{1 \cdot 2 \cdot 3} + \dfrac{5}{2 \cdot 3 \cdot 4} + \dfrac{6}{3 \cdot 4 \cdot 5} + \cdots + \dfrac{n + 3}{n(n + 1)(n + 2)}$
$= \dfrac{n(5n + 11)}{4(n + 1)(n + 2)}$.

19. $\dfrac{3}{1 \cdot 2 \cdot 2} + \dfrac{4}{2 \cdot 3 \cdot 2^2} + \dfrac{5}{3 \cdot 4 \cdot 2^3} + \cdots + \dfrac{n + 2}{n(n + 1) \cdot 2^n}$
$= 1 - \dfrac{1}{(n + 1) \cdot 2^n}$.

20. $1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} < 2 \cdot 1 - \frac{1}{n}$.
21. $x^n - y^n$ is divisible by $x - y$.
22. The sum of the interior angles of a polygon of n sides is $(n - 2) \cdot 180°$.
23. The number of diagonals in a polygon of n sides is $\frac{1}{2}n(n - 3)$.
24. The number of lines formed by joining n points, no three of which lie in the same straight line is $\frac{1}{2}n(n - 1)$.

Prove or disprove that the following are general formulas:

25. $1 + 2 + 3 + \cdots + n = \frac{1}{2}(2n^3 - 11n^2 + 23n - 12)$.
26. $1 + 8 + 27 + \cdots + n^3 = \frac{1}{2}(6n^3 - 17n^2 + 25n - 12)$.
27. $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n + 1)$
$= \frac{1}{3}n(n + 1)(n + 2)$.
28. $2 + 4 + 6 + \cdots + 2n = n^4 - 10n^3 + 36n^2 - 49n + 24$.
29. $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n(n + 1)} = \frac{n}{n + 1}$.
30. $\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \cdots + \frac{1}{(2n - 1)(2n + 1)} = \frac{n}{2n + 1}$.
31. $2 + 2^2 + 2^3 + \cdots + 2^n = 2^{n+1} - 2$.
32. $1 \cdot 4 + 4 \cdot 7 + 7 \cdot 10 + \cdots + (3n - 2)(3n + 1)$
$= n(3n^2 + 3n - 2)$.
33. $1^2 \cdot 2 + 2^2 \cdot 3 + 3^2 \cdot 4 + \cdots + n^2(n + 1)$
$= \frac{1}{12}n(n + 1)(n + 2)(3n + 1)$.
34. $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} < 3 - \frac{1}{n}$.
35. $n^2 - n + 41$ is a prime number

76. The binomial formula.

By the method of multiplying polynomials we find

$$(a + b)^1 = a + b,$$
$$(a + b)^2 = a^2 + 2ab + b^2,$$
$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3,$$
$$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.$$

If n represents the exponent of the binomial $a + b$ in any of the above expansions, we see that

I. *The first term is a^n.*
II. *The second term is $na^{n-1}b$.*
III. *The exponents of a decrease by 1 from term to term; the exponents of b increase by 1 from term to term.*
IV. *If in any term the coefficient is multiplied by the exponent of a and divided by the number of the term, the result is the coefficient of the next term.*

We therefore write

$$(a + b)^n = a^n + \frac{n}{1} a^{n-1}b + \frac{n(n-1)}{1 \cdot 2} a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} a^{n-3}b^3 + \cdots + b^n. \quad (1)$$

This is the **binomial formula.** It holds when n is a positive integer, as will be proved in section 79, and under certain restrictions, for other values of n.

Other properties of the expansion to be noted are:

V. *The sum of the exponents of a and b is n.*
VI. *There are $n + 1$ terms.*
VII. *The coefficients are symmetric* (e.g., the coefficient of the third term from the beginning is the same as that of the third term from the end).

EXERCISES X. B

Assuming that the formula holds, expand the following binomials:

1. $(x + y)^5$.
2. $(a - b)^6$.
3. $(1 + x)^6$.
4. $(x - 1)^6$.
5. $(a + b)^7$.
6. $(a + b)^8$.
7. $(a - b)^8$.
8. $(a - b)^9$.
9. $(x + y)^{10}$.

Expand and simplify:

10. $\left(2x^3 - \frac{y}{3}\right)^4$.

SOLUTION.

$$\left(2x^3 - \frac{y}{3}\right)^4 = (2x^3)^4 + 4(2x^3)^3\left(-\frac{y}{3}\right) + 6(2x^3)^2\left(-\frac{y}{3}\right)^2$$
$$+ 4(2x^3)\left(-\frac{y}{3}\right)^3 + \left(-\frac{y}{3}\right)^4 = 16x^{12} - \frac{32}{3}x^9y + \frac{8}{3}x^6y^2$$
$$- \frac{8}{27}x^3y^3 + \frac{y^4}{81}.$$

NOTE. In problems of this type *do not attempt to simplify before completing the expansion.* In simplifying, always divide out factors common to numerator and denominator.

11. $(2x + 3y)^3$. **12.** $(5a^2 - 2b)^3$. **13.** $(2x^2 + \sqrt{y})^4$.
14. $(3a^{1/2} + 4b^3)^5$. **15.** $(\frac{1}{2}x^3 - 3y^2)^6$. **16.** $\left(\frac{a^2}{3} + \frac{3}{b^3}\right)^6$.
17. $(xy^2 - 3z^3)^6$. **18.** $(0.9x + 0.1y)^4$. **19.** $(0.2x + 0.8y)^5$.

Find the first four terms of the following expansions:

20. $(a + b)^{18}$. **21.** $(x + y)^{25}$. **22.** $(1 + x)^{50}$.
23. Expand $(x + y - z)^3$.

SUGGESTION. First consider $(x + y)$ as a single quantity.

24. Find, by using the binomial formula, the value of $(0.99)^4$.

SUGGESTION. $0.99 = 1 - 0.01$.

Find, by using the binomial formula, the value of

25. $(3.2)^4$. **26.** $(1.2)^5$. **27.** $(103)^4$.
28. $(0.99)^5$. **29.** $(1.03)^5$. **30.** $(98)^5$.
31. $(999)^3$. **32.** $(998)^3$. **33.** $(101)^6$.

Find, by using the binomial formula, approximate values of the following:

34. $(1.1)^{10}$, correct to two decimal places.
35. $(1.01)^{15}$, correct to four decimal places.
36. $(0.99)^{15}$, correct to four decimal places.
37. $(1.02)^{20}$, correct to four decimal places.
38. $(1.003)^{25}$, correct to six decimal places.

77. Pascal's triangle.

The coefficients of the binomial formula can be obtained by a clever scheme known as **Pascal's triangle,** exhibited below. This triangle of numbers, bordered by 1's, can be formed by adding together any two adjacent numbers in a given row to obtain the number between them in the row next below. The number 1 at the vertex of the triangle may be thought of as the expansion of $(a + b)^0 = 1$, the second row gives the coefficients of $(a + b)^1$, the third row gives the coefficients of $(a + b)^2 = a^2 + 2ab + b^2$, and so on.

$$
\begin{array}{ccccccccccccccccc}
&&&&&&&&1&&&&&&&&\\
&&&&&&&1&&1&&&&&&&\\
&&&&&&1&&2&&1&&&&&&\\
&&&&&1&&3&&3&&1&&&&&\\
&&&&1&&4&&6&&4&&1&&&&\\
&&&1&&5&&10&&10&&5&&1&&&\\
&&1&&6&&15&&20&&15&&6&&1&&\\
&1&&7&&21&&35&&35&&21&&7&&1&\\
1&&8&&28&&56&&70&&56&&28&&8&&1\\
.&&.&&.&&.&&.&&.&&.&&.&&.
\end{array}
$$

78. The general term of the binomial formula.

By examining (1) of section 76 we may see that the term of the binomial formula involving b^r is

$$\frac{n(n-1)(n-2)\cdots \text{to } r \text{ factors}}{1 \cdot 2 \cdot 3 \cdots r} a^{n-r}b^r, \tag{1}$$

which may also be written

$$\frac{n(n-1)(n-2)\cdots(n-r+1)}{r\,!} a^{n-r}b^r, \tag{2}$$

where the symbol $r\,!$, called r **factorial,** is defined, for r a positive integer, as

$$r\,! = r(r-1)(r-2)\cdots 3\cdot 2\cdot 1. \qquad (3)$$

For example, $4\,! = 4\cdot 3\cdot 2\cdot 1.$

Actually, the term involving b^r is term number $r+1$, since b appears in the second term, b^2 in the third, b^3 in the fourth, and so on. The rth term is

$$\frac{n(n-1)(n-2)\cdots \text{to } r-1 \text{ factors}}{1\cdot 2\cdot 3\cdots (r-1)}a^{n-r+1}b^{r-1}. \qquad (4)$$

Either term number $r+1$ (formula (1) or (2)) or term number r (4) may be called the **general term** of the expansion of $(a+b)^n$.

EXERCISES X. C

1. Find the 4th term of $(2x - \frac{1}{4}y)^{10}$.

   SOLUTION. The 4th term will involve the 3rd power of $-\frac{1}{4}y$; it is

   $$\frac{10\cdot 9\cdot 8}{1\cdot 2\cdot 3}(2x)^7(-\tfrac{1}{4}y)^3 = -240x^7y^3.$$

2. Find the 6th term of $(x+y)^{15}$.
3. Find the 5th term of $(3x^2+2y^3)^{10}$.
4. Find the 7th term of $(x^3-2y^2)^{18}$.
5. Find the 8th term of $(5a^{1/2} - \frac{1}{2}b^5)^{12}$.
6. Find the next to the last term of $(7x^2-y^7)^{17}$.
7. Find the middle term of $(3a^4+4b^3)^8$.
8. Find the middle term of $\left(\dfrac{\sqrt{x}}{2}+\dfrac{\sqrt{2}}{y^2}\right)^{12}$.
9. Find the term involving b^5 in $(a-b)^{15}$.
10. Find the term involving a^5 in $(a-b)^{12}$.
11. Find the term involving y^8 in $(3x^3-2y^2)^7$.
12. Find the term involving x^{12} in $(2x^3-3y^2)^7$.

13. Find the term involving x^2y^6 in $(x + y - z)^{10}$.
14. Find the term involving b^4c^9 in $(a + b^2 + c^3)^8$.
15. The middle term of $(x^2 + 2y)^n$ is $1120x^8y^4$. Find n.

79. The binomial theorem for positive integral exponents.

We shall now prove that the binomial formula holds when the exponent n is a positive whole number. This is called the **binomial theorem** for positive integral exponents. The proof will be effected by the process of mathematical induction, explained earlier in the chapter.

Part I. *The formula is true for certain values of* n, viz., $n = 1, 2, 3, 4$, as has already been seen.

Part II. *If it is true for* $n = k$, *it is true for* $n = k + 1$. Assume that it is true for $n = k$. Then,

$$(a + b)^k = a^k + ka^{k-1}b + \cdots + \underbrace{\frac{k(k-1)\cdots(k-r+2)}{1\cdot 2\cdots(r-1)}a^{k-r+1}b^{r-1}}_{\text{term number } r\ (= T_r)} + \underbrace{\frac{k(k-1)\cdots(k-r+1)}{1\cdot 2\cdots r}a^{k-r}b^r}_{\text{term number } r+1\ (= T_{r+1})} + \cdots + b^k.$$

Multiply both sides by $a + b$:

$$(a + b)^{k+1} = a^{k+1} + ka^kb + \cdots + \underbrace{\frac{k(k-1)\cdots(k-r+1)}{1\cdot 2\cdots r}a^{k-r+1}b^r}_{(aT_{r+1})} + \cdots + a^kb + \cdots + \underbrace{\frac{k(k-1)\cdots(k-r+2)}{1\cdot 2\cdots(r-1)}a^{k-r+1}b^r}_{(bT_r)} + \cdots + b^{k+1}.$$

The coefficients of $a^{k-r+1}b^r$ in the terms aT_{r+1} and bT_r combine as follows:

$$\frac{k(k-1)\cdots(k-r+2)(k-r+1)}{1\cdot 2\cdots(r-1)r} + \frac{k(k-1)\cdots(k-r+2)}{1\cdot 2\cdots(r-1)}$$
$$= \frac{k(k-1)\cdots(k-r+2)}{1\cdot 2\cdots(r-1)r}(k-r+1+r)$$
$$= \frac{(k+1)k(k-1)\cdots(k-r+2)}{1\cdot 2\cdots r}.$$

Therefore,

$$(a+b)^{k+1} = a^{k+1} + (k+1)a^k b + \cdots + \frac{(k+1)k\cdots(k-r+2)}{1\cdot 2\cdots r}a^{k-r+1}b^r + \cdots + b^{k+1},$$

which is precisely what we should get by substituting $k+1$ for n in the binomial formula.

That is, if the formula is true for any value $n = k$, it is true for the next value, $n = k + 1$. But by Part I it is true for $n = 1, 2, 3, 4$, and consequently must be true for $n = 5, 6, \cdots$.

Therefore we may write, for any positive whole number n,

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2\,!}a^{n-2}b^2 + \cdots + \frac{n(n-1)\cdots(n-r+1)}{r\,!}a^{n-r}b^r + \cdots + b^n. \quad (1)$$

80. The binomial series.

If in the binomial formula we set $a = 1$, $b = x$, we obtain

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2\,!}x^2 + \cdots + \frac{n(n-1)\cdots(n-r+1)}{r\,!}x^r + \cdots. \quad (1)$$

If n is a positive whole number, this expansion will terminate with x^n; otherwise it will continue indefinitely, in which case it is called the **binomial series.** It can be shown that if x is numerically less than 1 $(-1 < x < 1)$, formula (1) holds for any real value of n, in the sense that as we include more and more terms on the right, we get closer and closer to the value on the left. (See Chapter XX.)

Example 1.

(a) Find the first four terms of (1) when $n = -2$. (b) Substitute $x = \frac{1}{4}$ and evaluate.

SOLUTION.

(a) $$(1 + x)^{-2} = 1 - 2x + \frac{-2(-3)}{1 \cdot 2}x^2 + \frac{-2(-3)(-4)}{1 \cdot 2 \cdot 3}x^3 + \cdots$$
$$= 1 - 2x + 3x^2 - 4x^3 + \cdots.$$

(b) $$1 - 2 \cdot \frac{1}{4} + 3 \cdot \frac{1}{16} - 4 \cdot \frac{1}{64} = \frac{5}{8} = 0.625.$$

The true value is $\left(1 + \frac{1}{4}\right)^{-2} = \left(\frac{5}{4}\right)^{-2} = \left(\frac{4}{5}\right)^2 = \frac{16}{25} = 0.64.$

Example 2.

Find the square root of 1.2 by using the first three terms of the binomial expansion.

SOLUTION.

$$\sqrt{1.2} = (1 + 0.2)^{1/2}$$
$$= 1 + \frac{1}{2}(0.2) + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{1 \cdot 2}(0.2)^2 + \cdots$$
$$= 1 + 0.1 - 0.005 + \cdots$$
$$= 1.095.$$

This is correct to three decimal places.

Any binomial $(a+b)^n$, *in which* $a \neq b$, *can be expanded.* Suppose, for definiteness, that a is numerically greater than b. Then

$$(a+b)^n = \left[a\left(1+\frac{b}{a}\right)\right]^n = a^n\left(1+\frac{b}{a}\right)^n$$

where b/a is numerically less than 1. The last binomial can be expanded by (1).

Example 3.

Find the square root of 23 by using the binomial formula.

SOLUTION.

$$\sqrt{23} = (25-2)^{1/2} = \left[25\left(1-\frac{2}{25}\right)\right]^{1/2} = 5(1-0.08)^{1/2}$$

$$= 5\left[1+\frac{1}{2}(-0.08)+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{1\cdot 2}(-0.08)^2+\cdots\right]$$

$$= 5(1-0.04-0.0008+\cdots) = 5(0.9592) = 4.796$$

which is correct to three decimal places.

EXERCISES X. D

Find the first four terms in the expansion of

1. $(1+x)^{-1}$.
2. $(1+x)^{-2}$.
3. $(1+x)^{1/2}$.
4. $(1-x)^{1/2}$.
5. $(1+x)^{-1/2}$.
6. $(1+2x)^{1/2}$.
7. $(1+x)^{1/3}$.
8. $(1+x)^{1/4}$.
9. $(1+x)^{1/5}$.
10. $(1+x)^{2/3}$.
11. $(1+x)^{0\cdot 4}$.
12. $(1-x^2)^{-1}$.

Find approximations to the square roots of the following numbers by using three terms of the binomial series:

13. 1.08.
14. 1.02.
15. 0.99.
16. 108.
17. 98.
18. 27.
19. 24.
20. 10.
21. 50.
22. 35.
23. 32.
24. 60.

Find approximations to the cube roots of the following numbers by using three terms of the binomial series:

25. 1.06. **26.** 0.99. **27.** 1003.
28. 999. **29.** 10. **30.** 7.
31. 28. **32.** 25. **33.** 128.

Find approximations to the fifth roots of the following numbers by using three terms of the binomial series:

34. 1.02. **35.** 0.95. **36.** 35.

CHAPTER XI

Progressions

81. Arithmetic progression.

An **arithmetic progression** is a sequence of quantities, called **terms,** each of which (after the first) can be obtained from the preceding by adding to it a fixed quantity called the **common difference.** Thus, the set of numbers 2, 5, 8, 11, for example, is an arithmetic progression. Here the common difference is $5 - 2 = 8 - 5 = 11 - 8 = 3$.

82. The *n*th term of an arithmetic progression.

Let us denote the arithmetic progression by

$$a_1, a_2, a_3, \cdots, a_n.$$

If the common difference is represented by d, we have, by definition,

$$a_2 = a_1 + d, \qquad a_3 = a_1 + 2d, \qquad a_4 = a_1 + 3d, \cdots,$$

and in general, for the nth term,

$$a_n = a_1 + (n - 1)d. \qquad (1)$$

Example.

Find the 10th term of the arithmetic progression 2, 5, 8, 11, $\cdots$.

SOLUTION. $a_{10} = 2 + 9 \cdot 3 = 2 + 27 = 29.$

83. The sum of an arithmetic progression.

Let us denote the sum of n terms of an arithmetic progression by S_n. To derive a formula for the sum, we write it first in natural order and then in reverse order:

$$S_n = a_1 + (a_1 + d) + (a_1 + 2d) + \cdots + (a_n - d) + a_n,$$
$$S_n = a_n + (a_n - d) + (a_n - 2d) + \cdots + (a_1 + d) + a_1.$$

Adding, we get

$$2S_n = (a_1 + a_n) + (a_1 + a_n) + (a_1 + a_n) + \cdots + (a_1 + a_n) + (a_1 + a_n) = n(a_1 + a_n)$$

(since there are n terms). Therefore,

$$S_n = \frac{n}{2}(a_1 + a_n). \quad (1)$$

If in (1) we substitute $a_n = a_1 + (n - 1)d$, as given by equation (1) of the preceding section, we obtain another useful expression for the sum, namely,

$$S_n = \frac{n}{2}[2a_1 + (n - 1)d]. \quad (2)$$

Example 1.

Find the sum of 10 terms of the arithmetic progression 2, 5, 8, $\cdots$.

SOLUTION. Substitute $a_{10} = 29$ (found above) in (1):

$$S_{10} = \frac{10}{2}(2 + 29) = 155.$$

Example 2.

Find the sum of 9 terms of the arithmetic progression 25, 21, 17, $\cdots$.

SOLUTION. $a_1 = 25$, $d = -4$, $n = 9$. Substitute in (2):

$$S_9 = \frac{9}{2}[2 \cdot 25 + 8(-4)] = 81.$$

84. Arithmetic means.

The terms between any two given terms of an arithmetic progression are called **arithmetic means** between the given terms.

Example.

Insert 3 arithmetic means between 7 and 13.

SOLUTION. $a_1 = 7$, $a_5 = 13$ ($n = 5$, since the progression is composed of the 3 means and the first and last terms). Substitute in formula (1) of section 82:

$$13 = 7 + 4d, \qquad d = 1\tfrac{1}{2}.$$

The progression is 7, $8\frac{1}{2}$, 10, $11\frac{1}{2}$, 13.

EXERCISES XI. A

Find a_n and S_n for the arithmetic progressions in exercises 1–10:

1. 3, 8, 13, $\cdots$ ($n = 10$).
2. 32, 29, 26, $\cdots$ ($n = 10$).
3. 2, 11, 20, $\cdots$ ($n = 25$).
4. $1\frac{1}{2}$, 4, $6\frac{1}{2}$, $\cdots$ ($n = 13$).
5. 11, 8, 5, $\cdots$ ($n = 12$).
6. 4, 13, 22, $\cdots$ ($n = 50$).
7. 0.5, 1.1, 1.7, $\cdots$ ($n = 100$).
8. 2, 9, 16, $\cdots$ ($n = 250$).
9. $\frac{1}{6}, \frac{1}{4}, \frac{1}{3}, \cdots$ ($n = 25$).
10. $\frac{1}{10}, \frac{1}{12}, \frac{1}{15}, \cdots$ ($n = 31$).
11. Given $a_1 = 3$, $d = 2\frac{1}{2}$, $a_n = 33$; find n and S_n.
12. Given $a_1 = \frac{3}{2}$, $d = \frac{3}{4}$, $a_n = 12$; find n and S_n.
13. Given $a_1 = 7$, $a_9 = 17$; find d and S_9.
14. Insert seven arithmetic means between 6 and 12.
15. Insert eight arithmetic means between 13 and 25.
16. Given $a_{10} = 47$, $d = 4$; find a_1 and S_{10}.
17. Given $a_{75} = 200$, $d = \frac{5}{6}$; find a_1 and S_{75}.
18. Given $a_1 = 13$, $S_9 = 171$; find d and a_9.
19. Given $a_1 = 30$, $S_7 = 140$; find d and a_7.
20. Given $a_{10} = 25$, $S_{10} = 30$; find a_1 and d.
21. Given $d = 6$, $S_8 = 76$; find a_1 and a_8.
22. Given $d = 1\frac{1}{2}$, $S_{11} = 176$; find a_1 and a_{11}.
23. Given $a_1 = 3$, $d = 7$, $S_n = 279$; find n and a_n.
24. Given $a_1 = 138$, $d = -4$, $S_n = 1392$; find n and a_n.

25. Given $a_1 = 21\frac{1}{2}$, $d = -1\frac{1}{4}$, $S_n = 124\frac{1}{4}$; find n and a_n.
26. Given $a_1 = 6$, $a_n = 60$, $S_n = 165$; find n and d.
27. Given $a_1 = 7$, $a_n = 39$, $S_n = 299$; find n and d.
28. Given $a_n = 152$, $d = 5\frac{1}{2}$, $S_n = 2150$; find n and a_1.
29. Given $a_4 = 26$, $a_{11} = 110$; find a_8.
30. Given $a_{37} = 232$, $a_{41} = 250$; find a_{25}.
31. Given $a_8 = 25$, $S_8 = 116$; find a_{10}.
32. Determine x so that x, $\frac{1}{2}x + 7$, $3x - 1$ will be an arithmetic progression.
33. Determine x so that $2x$, $3x + 1$, $x^2 + 2$ will be an arithmetic progression.
34. Derive a formula for the sum of the first n integers.
35. Derive a formula for the sum of the first n even integers.
36. Derive a formula for the sum of the first n odd integers.

Derive formulas for

37. a_1 and S_n in terms of d, n, a_n.
38. d and a_n in terms of a_1, n, S_n.
39. a_1 and d in terms of n, a_n, S_n.
40. a_1 and a_n in terms of d, n, S_n.
41. d and n in terms of a_1, a_n, S_n.
42. n and S_n in terms of a_1, d, a_n.
43. d in terms of a_1, n, a_n.
44. Prove formula (2) of section 83 by mathematical induction.

85. Geometric progression.

A **geometric progression** is a sequence of quantities, called **terms**, each of which (after the first) can be obtained from the preceding by multiplying it by a fixed quantity called the **common ratio.** The quantities 2, 6, 18, 54, for example, form a geometric progression. Here the common ratio is $\frac{6}{2} = \frac{18}{6} = \frac{54}{18} = 3$.

86. The *n*th term of a geometric progression.

If the geometric progression is

$$a_1, a_2, a_3, \cdots, a_n,$$

and the common ratio is r, it follows from the definition that

$$a_2 = a_1 r, \qquad a_3 = a_1 r^2, \qquad a_4 = a_1 r^3, \qquad \cdots,$$

and in general,

$$a_n = a_1 r^{n-1}. \tag{1}$$

Example.

Find the 5th term of the geometric progression 2, 6, 18, $\cdots$.

SOLUTION. The ratio is $\frac{6}{2}$ or $\frac{18}{6} = 3$. Therefore,

$$a_5 = 2 \cdot 3^4 = 162.$$

87. The sum of a geometric progression.

Denoting the sum of n terms of a geometric progression by S_n, we write

$$S_n = a_1 + a_1 r + a_1 r^2 + \cdots + a_1 r^{n-2} + a_1 r^{n-1}. \tag{1}$$

Multiply by r:

$$rS_n = a_1 r + a_1 r^2 + a_1 r^3 + \cdots + a_1 r^{n-1} + a_1 r^n. \tag{2}$$

Subtract (1) from (2):

$$rS_n - S_n = a_1 r^n - a_1,$$
$$(r - 1)S_n = a_1(r^n - 1).$$

Provided $r \neq 1$,

$$S_n = a_1 \frac{r^n - 1}{r - 1}. \tag{3}$$

If r is numerically less than 1, it is convenient to write

(3) in the form

$$S_n = a_1 \frac{1 - r^n}{1 - r}. \tag{4}$$

EXERCISE

Show that

$$S_n = \frac{ra_n - a_1}{r - 1} = \frac{a_1 - ra_n}{1 - r}. \tag{5}$$

Example.

Find the sum of 5 terms of the geometric progression 2, 6, 18, $\cdots$.

SOLUTION. By (3),

$$S_5 = 2 \cdot \frac{3^5 - 1}{3 - 1} = 242.$$

88. Geometric means.

The terms between any two given terms of a geometric progression are called **geometric means** between the given terms.

Example.

Insert 3 geometric means between 16 and 81.

SOLUTION. $a_1 = 16$, $a_5 = 81$. Substitute in the formula $a_n = a_1 r^{n-1}$:

$$81 = 16r^4, \qquad r^4 = \frac{81}{16}, \qquad r = \pm\frac{3}{2}.$$

($r = \pm\frac{3}{2}i$, also, but only real values will be considered.) There are two progressions:

$$16, 24, 36, 54, 81; \qquad 16, -24, 36, -54, 81.$$

EXERCISES XI. B

Find a_n and S_n for the following geometric progressions:

1. $4, 12, 36, \cdots (n = 6)$.
2. $5, 10, 20, \cdots (n = 10)$.
3. $6, -24, 96, \cdots (n = 8)$.
4. $32, 48, 72, \cdots (n = 7)$.
5. $156, 312, 624, \cdots (n = 11)$.
6. $-17, 34, -68, \cdots (n = 12)$.
7. $\frac{2}{3}, \frac{2}{5}, \frac{6}{25}, \cdots (n = 5)$.
8. $0.7, 0.07, 0.007, \cdots (n = 12)$.
9. $1, 1.03, (1.03)^2, \cdots (n = 5)$.
10. $\frac{1}{3}, -\frac{1}{6}, \frac{1}{12}, \cdots (n = 10)$.
11. $2, 2\sqrt{3}, 6, \cdots (n = 14)$.
12. $3, 3\sqrt{2}, 6, \cdots (n = 19)$.
13. $\sqrt{2}, \dfrac{1}{\sqrt{2}}, \dfrac{\sqrt{2}}{4}, \cdots (n = 6)$.
14. $2 + \sqrt{3}, 2, 8 - 4\sqrt{3}, \cdots (n = 5)$.
15. Given $a_1 = 8$, $r = 1\frac{1}{2}$, $a_n = 40\frac{1}{2}$; find n and S_n.
16. Given $a_1 = 567$, $r = \frac{1}{3}$, $a_n = \frac{7}{9}$; find n and S_n.
17. Given $a_1 = 80$, $a_5 = 405$; find r and S_5.

NOTE. Consider only real values in these exercises.

18. Given $a_1 = 9000$, $a_6 = 2.88$; find r and S_6.
19. Insert 3 geometric means between 5 and 12,005.
20. Insert 4 geometric means between 135 and $-17\frac{7}{9}$.
21. Given $a_5 = 625$, $r = \frac{5}{2}$; find a_1 and S_5.
22. Given $a_7 = 2$, $r = \frac{2}{3}$; find a_1 and S_7.
23. Given $r = 3$, $S_6 = 2548$; find a_1 and a_6.
24. Given $r = \frac{3}{4}$, $S_5 = 390\frac{1}{2}$; find a_1 and a_5.
25. Given $a_1 = 40$, $r = 0.2$, $S_n = 49.984$; find n and a_n.
26. Given $a_1 = \frac{9}{4}$, $r = -\frac{2}{3}$, $S_n = \frac{133}{108}$; find n and a_n.
27. Given $a_1 = 6$, $a_n = 1536$, $r = 2$; find n and S_n.
28. Given $a_1 = 5$, $r = 7$, $a_n = 588{,}245$; find n and S_n.
29. Given $a_1 = 162$, $a_n = 512$, $S_n = 1562$; find r and n.
30. Given $a_1 = 6$, $a_n = -162$, $S_n = -120$; find r and n.
31. Given $r = 4$, $a_n = 13{,}312$, $S_n = 17{,}732$; find a_1 and n.
32. Given $r = -\frac{3}{2}$, $a_n = 810$, $S_n = 550$; find a_1 and n.
33. Given $a_1 = 7$, $S_3 = 301$; find r and a_3.
34. Given $a_6 = 486$, $a_9 = 2250$; find a_{11}.
35. Given $a_5 = 12$, $a_9 = 108$; find a_4.
36. Given $a_5 = 3$, $a_{14} = 10$; find a_{23}.

SUGGESTION. Find r^9.

37. Determine x so that $x - 7$, $x + 5$, $2x$ will be a geometric progression.

38. Determine x so that $10 - x$, $x + 4$, $4x + 1$ will be a geometric progression.

Derive formulas for

39. r and S_n in terms of a_1, a_n, n.

40. a_1 and a_n in terms of r, n, S_n.

41. a_1 and S_n in terms of r, n, a_n.

42. a_n in terms of a_1, r, S_n.

43. a_1 in terms of r, a_n, S_n.

44. r in terms of a_1, a_n, S_n.

45. Prove formula (3) of section 87 by mathematical induction.

89. Infinite geometric progression.

Consider a line segment 1 unit in length. (Fig. 21.) Suppose that a point starts at one end of the segment and moves halfway to the other end, then half of the remaining distance, then half of the distance still remaining, and so on. The total distance traversed by the point will approach nearer and nearer to 1. But the distances through which the point moves are the terms of the geometric progression, $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots$, whose ratio is $\frac{1}{2}$. Thus, the sum of this progression, as we take more and more terms, approaches the value 1; that is, it can be made to differ from 1 by an amount as small as we please.

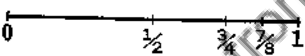


FIG. 21

A geometric progression in which the number of terms increases without limit is called an **infinite geometric progression** or **infinite geometric series.** If the ratio is numerically less than 1, the sum of n terms of the progression, as n increases without limit, approaches a perfectly definite value, called the **limit,** or **sum,** of the progression. (Observe that it is not a "sum" in the ordinary sense.)

For, from (4) of section 87, we see that

$$S_n = a_1 \frac{1 - r^n}{1 - r} = \frac{a_1}{1 - r} - \frac{a_1}{1 - r} r^n. \qquad (1)$$

Now if the ratio is numerically less than 1 ($|r| < 1$),* the numerical value of r^n decreases as n increases, and by taking n sufficiently large we can make $|r^n|$, and consequently $|a_1 r^n/(1 - r)|$, as small as we please. That is, as n increases without limit, the last fraction in (1) shrinks toward zero in value, and the sum of n terms of the progression approaches the value $a_1/(1 - r)$. In symbols, we write

$$S_\infty = \frac{a_1}{1 - r}. \tag{2}$$

Example.

Find the limit of the infinite geometric progression $2, \frac{2}{3}, \frac{2}{9}, \cdots$.

SOLUTION. $a_1 = 2, r = \frac{1}{3}$.

$$S_\infty = \frac{a_1}{1 - r} = \frac{2}{1 - \frac{1}{3}} = 3.$$

EXERCISES XI. C

Find the "sums" of the following infinite geometric progressions:

1. $2, \frac{2}{5}, \frac{2}{25}, \cdots$.
2. $4, 3, \frac{9}{4}, \cdots$.
3. $3, \frac{6}{7}, \frac{12}{49}, \cdots$.
4. $3, -\frac{6}{7}, \frac{12}{49}, \cdots$.
5. $-96, 24, -6, \cdots$.
6. $81, 54, 36, \cdots$.
7. $3, \sqrt{3}, 1, \cdots$.
8. $3, \frac{3\sqrt{2}}{2}, \frac{3}{2}, \cdots$.
9. $8\sqrt{5}, 10, \frac{5\sqrt{5}}{2}, \cdots$.
10. $\sqrt{6}, -\sqrt{2}, \frac{\sqrt{6}}{3}, \cdots$.
11. $2, 3, -\sqrt{5}, 7 - 3\sqrt{5}, \cdots$.
12. $2 - \sqrt{3}, 7 - 4\sqrt{3}, 26 - 15\sqrt{3}, \cdots$.
13. $0.4, 0.04, 0.004, \cdots$.
14. $4, 0.04, 0.0004, \cdots$.

* The symbol $|r|$, read "absolute value of r," means the value of r without regard to sign; e.g., $|2| = |-2| = 2$.

15. 0.4, −0.04, 0.004, $\cdots$. **16.** 0.7, 0.07, 0.007, $\cdots$.
17. 1, $(1.02)^{-1}$, $(1.02)^{-2}$, $\cdots$.
18. $(1.03)^{-1}$, $(1.03)^{-2}$, $(1.03)^{-3}$, $\cdots$.

90. Repeating decimals.

A repeating decimal is an infinite geometric progression. Thus, $0.444\cdots$ (sometimes written $0.\dot{4}$) is the progression

$$0.4 + 0.04 + 0.004 + 0.0004 + \cdots,$$

in which $a_1 = 0.4$, $r = 0.1$. Likewise, the repeating decimal $0.235,235,235\cdots$ (written $0.\dot{2}3\dot{5}$) is the progression

$$0.235 + 0.000,235 + 0.000,000,235 + \cdots,$$

in which $a_1 = 0.235$, $r = 0.001$.

Example 1.

Find the limiting value of $0.444\cdots$.

SOLUTION.

$$S_\infty = \frac{a_1}{1-r} = \frac{0.4}{1-0.1} = \frac{0.4}{0.9} = \frac{4}{9}.$$

This may be checked by dividing 4 by 9.

Example 2.

Find the limiting value of $0.7\dot{4}\dot{3}$.

SOLUTION. $0.7\dot{4}\dot{3} = 0.7 + 0.043 + 0.00043 + \cdots$. This is composed of 0.7 and a geometric progression in which $a_1 = 0.043$, $r = 0.01$. Thus,

$$\lim 0.7\dot{4}\dot{3} = \frac{7}{10} + \frac{0.043}{1-0.01} = \frac{7}{10} + \frac{0.043}{0.99}$$

$$= \frac{7}{10} + \frac{43}{990} = \frac{736}{990} = \frac{368}{495}.$$

Although the formula for S_∞ can be used to find the limiting value of a repeating decimal, perhaps the best method is that used in developing the formula for the sum of a geometric progression.

Example 3.

Find the limiting value of $0.444\cdots$.

SOLUTION. Let

$$x = 0.444\cdots. \qquad (1)$$

Multiply by 10: $$10x = 4.444\cdots. \qquad (2)$$

(2) − (1) $$9x = 4,$$

$$x = \frac{4}{9}.$$

Example 4.

Find the limiting value of $0.7\dot{4}\dot{3}$.

SOLUTION. Let $$x = 0.74343\cdots. \qquad (3)$$

Multiply by 100: $$100x = 74.34343\cdots. \qquad (4)$$

(4) − (3) $$99x = 73.6,$$

$$x = \frac{73.6}{99} = \frac{736}{990} = \frac{368}{495}.$$

EXERCISES XI. D

Find the limiting values of the following repeating decimals:

1. $0.666\cdots$. **2.** $0.555\cdots$. **3.** $0.777\cdots$.
4. $0.\dot{8}$. **5.** $0.3232\cdots$. **6.** $0.\dot{4}\dot{5}$
7. $2.3434\cdots$. **8.** $0.3444\cdots$. **9.** $0.24343\cdots$.
10. $0.\dot{1}2\dot{3}$. **11.** $0.12323\cdots$. **12.** $0.32121\cdots$.
13. $0.\dot{1}4285\dot{7}$. **14.** $0.\dot{4}2857\dot{1}$. **15.** $0.\dot{1}23\dot{4}$.
16. $0.999\cdots$. **17.** $0.9090\cdots$. **18.** $0.\dot{6}3\dot{6}$.

91. Harmonic progression.

A **harmonic progression** is a sequence of terms whose reciprocals are in arithmetic progression. For example, $\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}$ is a harmonic progression, since 2, 4, 6, 8 is an arithmetic progression.

Example 1.

Find the 7th term of the harmonic progression $\frac{1}{4}, \frac{2}{3}, -1, \cdots$.

SOLUTION. Form the corresponding arithmetic progression: $4, \frac{3}{2}, -1, \cdots$.

$$d = -2.5,$$
$$a_7 = a_1 + 6d = 4 + 6(-2.5) = -11.$$

Take reciprocal: $-\dfrac{1}{11}$.

The terms of a harmonic progression between any two given terms are called **harmonic means.**

Example 2.

Insert 4 harmonic means between $\frac{1}{2}$ and $\frac{1}{17}$.

SOLUTION. In the corresponding arithmetic progression,

$$a_1 = 2, \qquad a_6 = 17.$$
$$17 = 2 + 5d, \qquad d = 3.$$

Arithmetic progression: 2, 5, 8, 11, 14, 17.
Harmonic progression: $\frac{1}{2}, \frac{1}{5}, \frac{1}{8}, \frac{1}{11}, \frac{1}{14}, \frac{1}{17}$.

There is no simple method of finding the sum of a harmonic progression.

EXERCISES XI. E

1. Find the 20th term of the progression $\frac{1}{2}, \frac{1}{5}, \frac{1}{8}, \cdots$.
2. Find the 8th term of the progression 24, 12, 8, $\cdots$.
3. Find the 9th term of the progression $\frac{2}{3}, \frac{2}{5}, \frac{3}{10}, \cdots$.
4. Find the 10th term of the progression $\frac{1}{3}, -2, -\frac{1}{4}, \cdots$.
5. Insert 3 harmonic means between $\frac{1}{2}$ and $\frac{1}{4}$.
6. Insert 5 harmonic means between 3 and 12.
7. Insert 4 harmonic means between 10 and 60.
8. The 4th term of a harmonic progression is 15, the 10th term is 6. Find the 12th term.

9. The 4th term of a harmonic progression is 12, the 8th term is 6. What is the 16th term?
10. Determine x so that x, $x + 5$, $2x$ will be a harmonic progression.
11. Determine x so that $x + 1$, $x - 1$, $2x - 7$ will be a harmonic progression.
12. Derive a formula for the harmonic mean of two numbers a and b.

MISCELLANEOUS EXERCISES XI. F

1. Find a_{10} and S_{10} for the progression 16, 24, 32, $\cdots$.
2. Find a_8 and S_8 for the progression 16, 24, 36, $\cdots$.
3. Find a_5 and S_5 for the progression 21, $\frac{14}{3}$, $\frac{28}{27}$, $\cdots$.
4. Find a_8 and S_8 for the progression 5, $8\frac{5}{7}$, $12\frac{3}{7}$, $\cdots$.
5. Find a_{100} and S_{100} for the progression 17, 14.5, 12, $\cdots$.
6. Find a_7 and S_7 for the progression 3, -6, -15, $\cdots$.
7. Find a_7 and S_7 for the progression 3, -6, 12, $\cdots$.
8. Find the next term of the progression 6, 8, 12, $\cdots$.
9. Find a_9 for the progression 6, 4, 3, $\cdots$.
10. Insert (a) three arithmetic means, (b) three geometric means, (c) three harmonic means, between 8 and 24.
11. A particle sliding down a plane inclined at a certain angle travels 3 feet during the first second. In any second after the first it slides 6 feet farther than it did in the previous second. (a) How far does it slide in the 8th second? (b) How far does it slide in 8 seconds? (c) How long will it take to slide 300 feet?
12. How many numbers are there in the first 25 lines of Pascal's triangle? (See section 77.)
13. A swimmer trained by swimming 55 yards on the first day and, on each day after the first, twice as far as he did on the preceding day. On what day did he swim a mile?
14. A colony of bacteria is increasing at the rate of 50 per cent per hour. If there were originally 64,000 bacteria in the colony how many will there be at the end of 5 hours?
15. A rubber ball is dropped from a height of 27 feet. Each time that it hits the ground it bounces to a height $\frac{2}{3}$ of that from which it fell. (a) Find the distance that it travels up to the

time that it hits the ground for the 5th time. (b) Find the distance that it travels before coming to rest.

16. In a potato race 25 potatoes are placed on the ground 6 feet apart in a straight row. In line with the potatoes, and 25 feet from the first one, is placed a basket. A runner, starting from the basket, picks up the potatoes and carries them, one at a time, to the basket. Find the total distance that he runs.

17. For a cistern 6 feet in diameter filled with water, the amount of work done in pumping out the water is 882 foot-pounds for the first foot of depth, 2646 foot-pounds for the next foot, 4410 for the third foot, and so on. (That is, for any foot after the first the work done is 1764 foot-pounds more than for the preceding foot.) If the well is 20 feet deep how much work is done in pumping out all of the water?

18. When a vertical rectangular plate is submerged in water so that the upper edge is in the surface of the water, the pressure on the first foot of depth is $\frac{1}{2}lw$ pounds, where l is the horizontal dimension, in feet, of the plate and w is the weight of 1 cubic foot of water (approximately 62.4 pounds). The pressure on any foot after the first is lw pounds more than that on the preceding foot. Find the pressure on such a plate which measures 5 feet horizontally and 10 feet vertically.

19. A 20-gallon container is filled with pure acid. Five gallons are drawn off and replaced with water; then 5 gallons of the mixture are drawn off and replaced with water, and so on until 5 drawings and 5 replacements have been made. Find the amount of acid in the final mixture.

20. A drilling company contracts to drill a well at a cost of $1.50 for the first foot, $2.25 for the second foot, $3.00 for the third foot, and so on. How deep a well can be drilled for $2310?

21. A sheet of paper is 0.001 inch thick. It is cut in half and one part is placed on the other. The two pieces are again cut in half and the parts again stacked together. If this process is performed 15 times how thick will the stack be?

22. Ten 3-pound weights are placed 2 inches apart on a lever (whose weight can be neglected), the first weight being 6 inches from the fulcrum. How far from the fulcrum must a 30-pound weight be placed, on the other arm of the lever, to effect a balance?

23. The tickets in a lottery are numbered 1, 2, 3, and so on. The price of a ticket is the number of cents equal to the number on the ticket. There are 10 prizes, the pay-off being as follows: The first number drawn pays 25 cents and each number after the first pays twice as much as the one which precedes it. How many tickets must be sold in order that there will be no loss to the person conducting the lottery?

24. The side of a square is 16 inches long. A second square is formed by joining, in the proper order, the midpoints of the sides of the first square. A third square is formed by joining the midpoints of the sides of the second square, and so on. Find the area of the tenth square.

25. A square is inscribed in a circle of radius 6 inches. A circle is inscribed in the square, another circle in this second square, and so on. Find the sum of the areas of (a) all of the circles, (b) all of the squares.

26. A regular hexagon is inscribed in a circle of radius $3\frac{1}{2}$ inches. A second circle is inscribed in the hexagon, another hexagon in this second circle, and so on. Find the sum of the areas of (a) all of the circles, (b) all of the hexagons.

27. Two lines meet at a certain angle. From a point on the first line, 6 inches from the point of meeting, a perpendicular is dropped to the second line. From the foot of the perpendicular another perpendicular is dropped back to the first line, then another perpendicular is dropped to the second line, and so on. Find the sum of the lengths of all of the perpendiculars when the angle at which the given lines meet is (a) 45°, (b) 60°, (c) 30°.

28. The sum of three numbers in arithmetic progression is 60. If the numbers are increased by 2, 1, and 28, respectively, the new numbers will be in geometric progression. Find the arithmetic progression.

SUGGESTION. Let the arithmetic progression be $x - d$, x, $x + d$.

29. The sum of four numbers in arithmetic progression is 28, the sum of their squares is 276. Find the numbers.

30. The coefficients of the 5th, 6th, and 7th terms in the expansion of $(a + b)^n$ are in arithmetic progression. Find n.

31. Given two unequal positive numbers x and y. Denote their arithmetic, geometric, and harmonic means by A, G, H, respectively. Prove that (a) $AH = G^2$, (b) $A > G > H$.

SUGGESTION. $(\sqrt{x} - \sqrt{y})^2 > 0$.

32. If x^2, y^2, z^2, form an arithmetic progression, prove that $y + z$, $z + x$, $x + y$ form a harmonic progression.

33. The geometric and harmonic means of two numbers are, respectively, 9 and 5.4. Find the numbers.

CHAPTER XII

Complex Numbers

92. Imaginary and complex numbers.

In section 44, of which the present chapter is a review and a continuation, we introduced the imaginary unit i having the property $i^2 = -1$, assuming that this new number obeys all the laws of addition and multiplication that hold for real numbers. Since $i^3 = i^2 \cdot i = -i$, $i^4 = (i^2)^2 = 1$, $i^5 = i^4 \cdot i = i$, $\cdots$, it is seen that the successive integral powers of i run through the cycle i, -1, $-i$, 1.

A number of the form $a + bi$, in which a and b are real numbers, is called a **complex number.** The number a is called the **real part,** and bi is called the **imaginary part** of the complex number, b being the coefficient of the imaginary part. If $b \neq 0$, the complex number is called an **imaginary number.** If $b \neq 0$ and $a = 0$, the complex number reduces to the form bi, which is called a **pure imaginary number.** If $b = 0$, the complex number reduces to the real number a.

Thus, complex numbers include real numbers and imaginary numbers as special cases.

Examples.

Real numbers: $-\frac{1}{2}$, π, $7 - \sqrt{3}$, $\sqrt[3]{-5}$.

Imaginary numbers: $3 - 2i$, $3 - i\sqrt{2}$, $\sqrt{3} + \sqrt{-3}$, $3i$, $-3i$, $i\sqrt{3}$, $\sqrt{-5}$.

Pure imaginary numbers: $3i$, $-3i$, $i\sqrt{3}$, $\sqrt{-5}$.

All of these examples are complex numbers.

The two complex numbers $a + bi$ and $a - bi$, which differ only in the signs of their imaginary parts, are called **conjugate complex numbers.** Either is said to be the conjugate of the other.

Two complex numbers, $a + bi$ and $c + di$, are equal if and only if their real parts are equal and their imaginary parts are equal. In particular, $a + bi = 0$ if and only if $a = 0$ and $b = 0$.

93. Addition and subtraction of complex numbers.

By definition, addition or subtraction of complex numbers will be effected by adding or subtracting their real parts to obtain the real part of their sum or difference, and by adding or subtracting their imaginary parts to obtain the imaginary part of their sum or difference. Thus,

$$(a + bi) + (c + di) = (a + c) + (b + d)i,$$
$$(a + bi) - (c + di) = (a - c) + (b - d)i.$$

Example 1.

Add $2 + 3i$ and $4 - 7i$.

SOLUTION.

$$\begin{array}{r} 2 + 3i \\ \underline{4 - 7i} \\ \text{Sum} = 6 - 4i. \end{array}$$

Example 2.

Subtract $4 - 7i$ from $2 + 3i$.

SOLUTION.

$$\begin{array}{r} 2 + \ \ 3i \\ \underline{4 - \ \ 7i} \\ \text{Difference} = -2 + 10i. \end{array}$$

94. Multiplication of complex numbers.

By definition, we shall multiply complex numbers according to the laws for real numbers, simplifying results by making use of the relation $i^2 = -1$. Thus,

$$(a + bi)(c + di) = ac + adi + bci + bdi^2$$
$$= (ac - bd) + (ad + bc)i.$$

Example.

Multiply $2 + 3i$ by $4 - 7i$.

SOLUTION.

$$\begin{array}{l} 2 + 3i \\ \underline{4 - 7i} \\ 8 + 12i \\ \underline{ - 14i - 21i^2} \\ 8 - 2i - 21i^2 = 8 - 2i + 21 = 29 - 2i. \end{array}$$

95. Division of complex numbers.

Division of complex numbers can be accomplished by writing the quotient in fractional form and multiplying numerator and denominator by the conjugate of the denominator. Thus,

$$\frac{a + bi}{c + di} = \frac{a + bi}{c + di} \cdot \frac{c - di}{c - di} = \frac{ac - adi + bci - bdi^2}{c^2 - d^2i^2}$$

$$= \frac{(ac + bd) + (bc - ad)i}{c^2 + d^2}$$

$$= \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2} i \quad (c + di \neq 0).$$

Example.

Divide $2 + 3i$ by $4 - 7i$.

SOLUTION.

$$\frac{2+3i}{4-7i}\cdot\frac{4+7i}{4+7i}=\frac{8+26i+21i^2}{16-49i^2}$$
$$=\frac{8+26i-21}{16+49}=\frac{-13+26i}{65}=-\frac{1}{5}+\frac{2}{5}i.$$

EXERCISES XII. A

Perform the indicated operations and reduce the result to the form $a + bi$:

1. $(3 + 5i) + (7 + 4i)$.
2. $(4 - 8i) + (6 - 3i)$.
3. $(5 + 6i) - (8 + 3i)$.
4. $(3 + 7i) - (5 - i)$.
5. $(-5 + 8i) + (3 - 10i)$.
6. $(9 - 4i) - (-3 + 4i)$.
7. $5 - (13 - 7i)$.
8. $(\frac{2}{3} - \frac{3}{4}i) - (\frac{1}{2} + \frac{5}{6}i)$.
9. $(5 + 3i) + (2 - 5i) + (-7 + i)$.
10. $(-6 + 2i) - (3 - 8i) - (-1 + 7i)$.
11. $(3 + 2\sqrt{2}\cdot i) + (5 - 7\sqrt{2}\cdot i)$.
12. $(3\sqrt{3} - 5\sqrt{7}\cdot i) - (4\sqrt{3} + 2\sqrt{7}\cdot i)$.
13. $(5 + \sqrt{-2}) - (2 + 3\sqrt{-2})$.
14. $(2 - \sqrt{-3}) - (3 - \sqrt{-2})$.
15. $(3 + 5i)(7 + 4i)$.
16. $(4 - 8i)(6 - 3i)$.
17. $(5 + 6i)(8 + 3i)$.
18. $(3 + 7i)(5 - i)$.
19. $(-5 + 8i)(3 - 10i)$.
20. $(9 - 4i)(-3 + 4i)$.
21. $(\frac{2}{3} - \frac{3}{4}i)(\frac{1}{2} + \frac{5}{6}i)$.
22. $(0.2 - 0.3i)(0.5 + 0.6i)$.
23. $(5 + 3i)(2 - 5i)(-7 + i)$.
24. $(-6 + 2i)(3 - 8i)(-1 + 7i)$.
25. $(3 + 2\sqrt{2}\cdot i)(5 - 7\sqrt{2}\cdot i)$.
26. $(3\sqrt{3} - 5\sqrt{7}\cdot i)(4\sqrt{3} + 2\sqrt{7}\cdot i)$.
27. $(5 + \sqrt{-2})(2 + 3\sqrt{-2})$.
28. $(2 - \sqrt{-3})(3 - \sqrt{-2})$.
29. $(6 - 5i)^2$.
30. $(2 + 3i)^3$.
31. $(3 + 5i) \div (7 + 4i)$.
32. $(4 - 8i) \div (6 - 3i)$.
33. $(5 + 6i) \div (8 + 3i)$.
34. $(3 + 7i) \div (5 - i)$.
35. $(-5 + 8i) \div (3 - 10i)$.
36. $(9 - 4i) \div (-3 + 4i)$.
37. $(\frac{2}{3} - \frac{3}{4}i) \div (\frac{1}{2} + \frac{5}{6}i)$.
38. $(0.2 - 0.3i) \div (0.5 + 0.6i)$.
39. $(3 + 2\sqrt{2}\cdot i) \div (5 - 7\sqrt{2}\cdot i)$.

40. $(3\sqrt{3} - 5\sqrt{7}\cdot i) \div (4\sqrt{3} + 2\sqrt{7}\cdot i)$.

41. $(5 + \sqrt{-2}) \div (2 + 3\sqrt{-2})$.

42. $(2 - \sqrt{-3}) \div (3 - \sqrt{-2})$.

43. $(6 + 5i) \div (6 - 5i)$. **44.** $7i \div (3 + 2i)$.

45. $\dfrac{(5 + 4i)(7 - i)}{2 + 3i}$. **46.** $\dfrac{9 - 7i}{(6 + i)(3 - 2i)}$.

47. $\dfrac{(8 + 5i)(2 - i)}{(7 - 3i)(2 + 3i)}$. **48.** $\dfrac{(5 - 3i)(6 + 7i)}{(5 + 3i)(6 - 7i)}$.

49. Show that the sum of two conjugate imaginary numbers is a real number and that their difference is a pure imaginary number.

50. Show that the product of two conjugate imaginary numbers is real and positive.

96. Graphic representation of complex numbers.

The complex number $x + yi$ may be represented by the point P whose coordinates are (x, y). (See Fig. 22.) When complex numbers are so represented, the x-axis is called the **axis of real numbers** and the y-axis the **axis of imaginary numbers.** All real numbers lie on the axis of real numbers, all pure imaginary numbers lie on the axis of imaginary numbers.

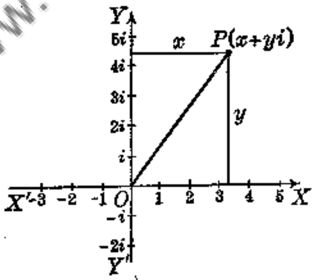


FIG. 22

The complex number $x + yi$ may also be represented by the line OP going from the origin to the point P. Such a line, having length and direction, is called a **vector.** Vectors are important in physics and mechanics.

97. Graphic addition and subtraction of complex numbers.

Let the complex numbers $a + bi$ and $c + di$ be represented by the points M and N respectively, and their sum,

$(a + c) + (b + d)i$, by the point P. (See Fig. 23.) Draw OM, ON, MP, NP; drop NQ, MR, PS perpendicular to OX; and draw MT perpendicular to PS. Then $MT = RS = OS - OR = (a + c) - a = c = OQ$, $TP = SP - ST = (b + d) - b = d = QN$. Therefore the right triangles MTP and OQN are congruent, and $MP = ON$. Also $\angle TPM = \angle QNO$ and MP is parallel to ON. Quadrilateral $OMPN$ is a parallelogram, since two sides are both equal and parallel.

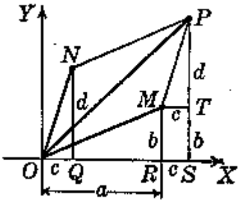


FIG. 23

Thus, to add two complex numbers graphically, or geometrically, *complete the parallelogram having as adjacent sides the lines drawn from the origin to the points representing the two numbers. The fourth vertex of this parallelogram will be the point representing the sum of the two numbers.**

If we think of the complex numbers $a + bi$ and $c + di$ as represented by the vectors OM and ON in Fig. 23, the sum of the numbers will be the vector OP.

To subtract $c + di$ from $a + bi$ graphically, we may add $a + bi$ and $-c - di$. Note that the negative of a complex number represented by the point P can be obtained by extending PO to a point P' such that OP and OP' are equal in length.

EXERCISES XII. B

Perform the indicated operations geometrically:

1. $(7 + 3i)(1 + 5i)$.
2. $(8 + i) + (5 - 4i)$.
3. $(5 + 2i) + (2 + 5i)$.
4. $(-6 + 3i) + (4 + 7i)$.
5. $(4i) + (3 + 2i)$.
6. $(3i) + (-2 + 4i)$.
7. $(8 - 9i) - (4 + 3i)$.
8. $(10 + 7i) - (3 - 4i)$.
9. $(-7 + 8i) - (13 - 2i)$.
10. $(10 + 5i) + (2 + i)$.
11. $(8 + 3i) + (8 - 3i)$.
12. $(8 - 3i) + (8 - 3i)$.

* If the origin O and the points M and N lie in the same straight line, the point P will lie in this straight line and OP will be equal to $OM + ON$.

13. $(8 + 3i) - (8 - 3i)$. **14.** $(7 + 2i) - (13 + 2i)$.
15. $(-7 + 3i) + (7 - 12i)$. **16.** $(10 + 2i) - (2 - 10i)$.
17. $(3 + 2i) + (1 + 8i) + (6 - 3i)$.

SUGGESTION. Combine the first two numbers geometrically and then combine the result with the third number.

18. $(6 + 7i) + (3 - 2i) - (-5 + 5i)$.
19. $(10 - 5i) - (2 - 7i) - (5 + 8i)$.
20. $(-2 + 3i) - (4 + 7i) + (6 + 4i)$.
21. Given the complex numbers $10 - 3i$, $5 + 2i$, $-3 + 8i$. Show that the same result is obtained by geometrically

(a) adding the first and second and then adding their sum to the third,
(b) adding the first and third and then adding their sum to the second,
(c) adding the second and third and then adding their sum to the first.

98. Trigonometric form of complex numbers.*

Let the complex number $x + yi$ be represented by the point P in Fig. 24. Let $OP = r$, and let the angle XOP be designated by A. Then

$$x = r \cos A, \qquad y = r \sin A,$$

and the complex number may be written

$$x + yi = r(\cos A + i \sin A). \quad (1)$$

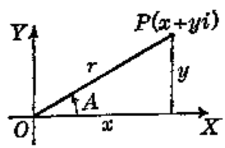


FIG. 24

This last form, which may be conveniently abbreviated r **cis** A, is called the **trigonometric,** or **polar, form** of the number. The angle A is called the **amplitude,** or **argument,** of the number, the length $r = \sqrt{x^2 + y^2}$ is called the **modulus,** or **absolute value,** of the number.

* The remaining sections of this chapter presuppose a knowledge of trigonometry.

Note that $\tan A = y/x$. For A we usually select the smallest angle satisfying the relation $0° \leqq A < 360°$.

The definition of absolute value of a complex number is consistent with the definition given for the absolute value of a real number. (See page 165, footnote.) The absolute value of $x + yi$, namely $\sqrt{x^2 + y^2}$, is also written $|x + yi|$.

EXERCISES XII. C*

Reduce to trigonometric form:

1. $-5 + 5i$.

SOLUTION. $r = \sqrt{(-5)^2 + 5^2} = 5\sqrt{2}$.

$$\tan A = \frac{5}{-5} = -1, \quad A = 135°.$$

$$-5 + 5i = 5\sqrt{2}\,(\cos 135° + i \sin 135°).$$

(The student is advised to plot the point as an aid in determining the angle.)

2. $3 + 4i$.

SOLUTION.

$$r = \sqrt{3^2 + 4^2} = 5.$$

$$\tan A = \frac{4}{3} = 1.3333,$$

$$A = 53.1° \text{ (from trigonometric tables).}$$

$$3 + 4i = 5\,(\cos 53.1° + i \sin 53.1°).$$

3. $3 + 3i$. **4.** $1 + i\sqrt{3}$. **5.** $4 - 3i$.
6. $4\sqrt{3} + 4i$. **7.** $5i$. **8.** -6.
9. $-5 - 5\sqrt{3}\cdot i$. **10.** $15 - 8i$. **11.** $-5 + 12i$.
12. $3 + 5i$. **13.** $-9i$. **14.** $7 - i$.
15. $-8 + 6i$. **16.** $-7 - 24i$. **17.** $20 + 21i$.
18. $\frac{1}{4} - \frac{1}{3}i$. **19.** $-\sqrt{2} + i\sqrt{7}$. **20.** $10 - 10i$.

* Table VIII at the back of the book may be used in solving these exercises.

Reduce to rectangular form (i.e. the form $a + bi$):

21. $6(\cos 30^\circ + i \sin 30^\circ)$.
22. $8(\cos 60^\circ + i \sin 60^\circ)$.
23. $4(\cos 45^\circ + i \sin 45^\circ)$.
24. $10(\cos 120^\circ + i \sin 120^\circ)$.
25. $3(\cos 180^\circ + i \sin 180^\circ)$.
26. $5(\cos 225^\circ + i \sin 225^\circ)$.
27. $2(\cos 210^\circ + i \sin 210^\circ)$.
28. $7(\cos 300^\circ + i \sin 300^\circ)$.
29. $\sqrt{2}(\cos 135^\circ + i \sin 135^\circ)$.
30. $\sqrt{3}(\cos 240^\circ + i \sin 240^\circ)$.
31. $\cos(-30^\circ) + i \sin(-30^\circ)$.
32. $10(\cos 65^\circ + i \sin 65^\circ)$.
33. $20(\cos 165^\circ + i \sin 165^\circ)$.
34. $5(\cos 265^\circ + i \sin 265^\circ)$.
35. $8(\cos 365^\circ + i \sin 365^\circ)$.
36. $7(\cos 1000^\circ + i \sin 1000^\circ)$.

99. Multiplication and division of complex numbers in trigonometric form.

Complex numbers expressed in trigonometric form can be multiplied by a very simple formula. For example,

$$\begin{aligned} &r_1(\cos A_1 + i \sin A_1) \cdot r_2(\cos A_2 + i \sin A_2) \\ &= r_1 r_2[\cos A_1 \cos A_2 - \sin A_1 \sin A_2 + i(\sin A_1 \cos A_2 \\ &\qquad + \cos A_1 \sin A_2)] \\ &= r_1 r_2[\cos (A_1 + A_2) + i \sin (A_1 + A_2)]. \qquad (1) \end{aligned}$$

That is, *the product of two complex numbers is a complex number whose absolute value is the product of their absolute values and whose amplitude is the sum of their amplitudes.*

Division is the inverse of multiplication, and it can readily be shown that, inversely, *the quotient of two complex numbers is a complex number whose absolute value is the absolute value of the dividend divided by the absolute value of the divisor and whose amplitude is the amplitude of the dividend minus the amplitude of the divisor.* That is,

$$\frac{r_1(\cos A_1 + i \sin A_1)}{r_2(\cos A_2 + i \sin A_2)} = \frac{r_1}{r_2}[\cos (A_1 - A_2) + i \sin (A_1 - A_2)]. \qquad (2)$$

EXERCISES XII. D

Perform the indicated operations, first reducing the numbers to trigonometric form (when necessary):

1. $2(\cos 50° + i \sin 50°) \cdot 3(\cos 20° + i \sin 20°)$.
2. $4(\cos 65° + i \sin 65°) \cdot 5(\cos 35° + i \sin 35°)$.
3. $7(\cos 100° + i \sin 100°) \cdot 2(\cos 200° + i \sin 200°)$.
4. $(\sqrt{3} - i)(3 + 3i)$.
5. $(-2 + 2i)(5\sqrt{3} + 5i)$.
6. $8(\cos 85° + i \sin 85°) \div 2(\cos 50° + i \sin 50°)$.
7. $6(\cos 100° + i \sin 100°) \div 3(\cos 230° + i \sin 230°)$.
8. $(-4 + 4i) \div (1 + i\sqrt{3})$.
9. $(-6\sqrt{3} + 6i) \div (\sqrt{2} - i\sqrt{6})$.
10. $(1 + i) \div (1 - i)$.
11. $(-3 - i\sqrt{3}) \div (\sqrt{6} - 3\sqrt{2} \cdot i)$.

100. Powers of complex numbers.

Raising to a power is a special case of multiplication, and it follows by a repeated application of (1) of the preceding section that

$$[r(\cos A + i \sin A)]^n = r^n(\cos nA + i \sin nA), \qquad (1)$$

where n is a positive integer.

The relation (1) is known as **De Moivre's Theorem.**

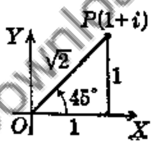


FIG. 25

Example.

Find the value of $(1 + i)^5$.

SOLUTION. Plot the number $1 + i$. The absolute value is $\sqrt{2}$ and the amplitude is 45°.

$$\begin{aligned}(1 + i)^5 &= [\sqrt{2}(\cos 45° + i \sin 45°)]^5 \\ &= 4\sqrt{2}(\cos 5 \cdot 45° + i \sin 5 \cdot 45°) \\ &= 4\sqrt{2}(\cos 225° + i \sin 225°) = 4(-1 - i).\end{aligned}$$

The student may check this result by expanding $(1 + i)^5$ by the binomial theorem.

101. Roots of complex numbers.

To prove De Moivre's Theorem for the case in which the exponent is the reciprocal of a positive integer, take the expression

$$[r(\cos A + i \sin A)]^{1/n} = r^{1/n}(\cos A + i \sin A)^{1/n}. \quad (1)$$

Let $A = nB$. Then the right side of (1) reduces to

$$r^{1/n}(\cos nB + i \sin nB)^{1/n} = r^{1/n}[(\cos B + i \sin B)^n]^{1/n}$$
$$= r^{1/n}(\cos B + i \sin B),$$

or

$$[r(\cos A + i \sin A)]^{1/n} = r^{1/n}\left(\cos \frac{A}{n} + i \sin \frac{A}{n}\right). \quad (2)$$

Since for any whole number k,

$$\cos (A + k \cdot 360°) = \cos A, \quad \sin (A + k \cdot 360°) = \sin A,$$

we have

$$[r(\cos A + i \sin A)]^{1/n}$$
$$= [r\{\cos (A + k \cdot 360°) + i \sin (A + k \cdot 360°)\}]^{1/n}$$
$$= r^{1/n}\left(\cos \frac{A + k \cdot 360°}{n} + i \sin \frac{A + k \cdot 360°}{n}\right). \quad (3)$$

By giving values to k from 0 to $n - 1$ inclusive, we obtain n distinct roots of the number $r(\cos A + i \sin A)$. If we give k any other positive integral value we get one of the n numbers already obtained. Therefore any complex number $r(\cos A + i \sin A)$ has exactly n distinct nth roots.

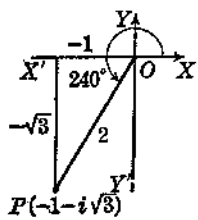


FIG. 26

Example.

Find the fourth roots of $-1 - i\sqrt{3}$.

SOLUTION. Plot the number $-1 - i\sqrt{3}$ and note that

$$-1 - i\sqrt{3} = 2(\cos 240° + i \sin 240°),$$

$$(-1 - i\sqrt{3})^{1/4} = 2^{1/4}\left(\cos\frac{240° + k \cdot 360°}{4} + i \sin\frac{240 + k \cdot 360°}{n}\right)$$

$$= \sqrt[4]{2}[\cos(60° + k \cdot 90°) + i \sin(60° + k \cdot 90°)].$$

Giving k successively the values 0, 1, 2, 3, we find for the four distinct fourth roots of $-1 - i\sqrt{3}$:

$$\sqrt[4]{2}(\cos 60° + i \sin 60°) = \sqrt[4]{2}\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = \frac{1}{2}\sqrt[4]{2} + \frac{i}{2}\sqrt[4]{18},$$

$$\sqrt[4]{2}(\cos 150° + i \sin 150°) = \sqrt[4]{2}\left(-\frac{\sqrt{3}}{2} + \frac{i}{2}\right)$$

$$= -\frac{1}{2}\sqrt[4]{18} + \frac{i}{2}\sqrt[4]{2},$$

$$\sqrt[4]{2}(\cos 240° + i \sin 240°) = \sqrt[4]{2}\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$$

$$= -\frac{1}{2}\sqrt[4]{2} - \frac{i}{2}\sqrt[4]{18},$$

$$\sqrt[4]{2}(\cos 330° + i \sin 330°) = \sqrt[4]{2}\left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right) = \frac{1}{2}\sqrt[4]{18} - \frac{i}{2}\sqrt[4]{2}.$$

In Fig. 27, the point P represents the complex number $2(\cos 240° + i \sin 240°)$; P_1, P_2, P_3, P_4 represent the four roots whose amplitudes are 60°, 150°, 240°, 330°, respectively.

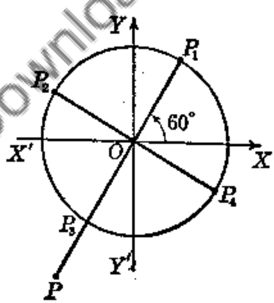


FIG. 27

Note that the roots can be found graphically as follows: Draw a circle with center at the origin and with a radius equal to the numerical fourth root of the absolute value of the number whose fourth roots are to be found; that is, a radius equal to $\sqrt[4]{2}$. Take one-fourth of the amplitude of the original number ($\frac{1}{4} \cdot 240° = 60°$). This locates the point P_1 on the circle. The four roots all lie on the circle and are spaced at equal intervals of 90°. Thus we can find P_2, P_3, P_4.

In general, the nth roots of the complex number $r(\cos A + i \sin A)$ can be found as follows: *Draw a circle whose center is the origin and whose radius is the numerical nth root of r; divide the angle A by n, the index of the root. Now divide the circumference of the circle, from A/n to $A/n + 360°$, into n equal parts. The n points of division will be the required roots.*

EXERCISES XII. E

Use De Moivre's Theorem to raise to the indicated powers:

1. $[5(\cos 16° + i \sin 16°)]^3$.
2. $[\sqrt{5}(\cos 22° + i \sin 22°)]^6$.
3. $(1 + i\sqrt{3})^5$.
4. $(1 - i)^8$.
5. $[\sqrt{3}(\cos 50° + i \sin 50°)]^{10}$.
6. $[\cos(-50°) + i \sin(-50°)]^{12}$.
7. $\left(-\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)^3$.
8. $\left(\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)^3$.
9. $(-\sqrt{2} + i\sqrt{2})^8$.
10. $(1 + i)^{-7}$.
11. $[2(\cos 50° + i \sin 50°)]^{-4}$.
12. $[\cos(-50°) + i \sin(-50°)]^{-4}$.

Find all of the

13. Square roots of $25(\cos 40° + i \sin 40°)$.
14. Square roots of $9(\cos 200° + i \sin 200°)$.
15. Square roots of $36(\cos 340° + i \sin 340°)$.
16. Cube roots of $8(\cos 36° + i \sin 36°)$.
17. Cube roots of $216(\cos 216° + i \sin 216°)$.
18. Square roots of $-\sqrt{3} + i$.
19. Square roots of $1 + i$.
20. Cube roots of $-\sqrt{3} + i$.
21. Cube roots of $-1 - i$.
22. Square roots of i.
23. Cube roots of 1.

SUGGESTION. $1 = \cos 0° + i \sin 0°$.

24. Fifth roots of 1.

25. Fifth roots of $4 + 4i$.

26. Fifth roots of $\frac{1}{2} - \frac{\sqrt{3}}{2} i$.

27. Fourth roots of $-i$.

28. Cube roots of $-7 + 6i$.

29. Sixth roots of $25i$.

30. Seventh roots of $-64(\sqrt{3} + i)$.

Obtain all of the roots of the following equations:

31. $x^3 - 216 = 0$. **32.** $x^5 + 1 = 0$. **33.** $x^4 + 64 = 0$.

34. $x^4 + x^2 + 1 = 0$. **35.** $x^4 - x^2 + 1 = 0$. **36.** $x^8 - x = 0$.

37. $x^4 + x^3 + x^2 + x + 1 = 0$.

Suggestion. Multiply by $x - 1$, solve the resulting equation, and discard the extraneous root $x = 1$.

38. $x^4 - x^3 + x^2 - x + 1 = 0$.

CHAPTER XIII

Theory of Equations

102. Polynomial.

A **polynomial** of degree n in the variable x is the function

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n, \qquad (1)$$

in which n is a positive integer, and the a's are constants, of which $a_0 \neq 0$. For example,

$$2x^5 - 3x^4 + 7x^3 - x + 10$$

is a polynomial of degree 5. The polynomial (1) is also called an **integral rational function** of degree n. (Cf. sections 8 and 18.) If we set it equal to zero we have an **integral rational equation** of degree n. A root of the equation $f(x) = 0$ is called a **zero** of the function $f(x)$.*

Although the coefficients in (1) may be any complex numbers, we shall restrict our discussion to polynomials and integral rational equations having real coefficients. However, in any theorem and its proof, the coefficients may be complex unless otherwise stated.

103. Remainder theorem.

If a polynomial $f(x)$ is divided by $x - r$, the remainder is equal to the value of the polynomial when r is substituted for x.

Divide the polynomial by $x - r$ until the remainder, which may be zero, is independent of x. Denote the quotient by $Q(x)$ and the remainder by R. Then, according to

* This definition is not restricted to integral rational functions.

the meaning of division,

$$f(x) \equiv (x - r)Q(x) + R.$$

Since this is an identity in x, it is satisfied by all values of x, and if we set $x = r$ we find that

$$f(r) = (r - r)Q(r) + R = 0 \cdot Q(r) + R = R.$$

Here it is assumed that a polynomial is finite for every finite value of the variable. Consequently, since $Q(x)$ is a polynomial, $Q(r)$ is a finite number, and $0 \cdot Q(r) = 0$.

Example.

$$
\begin{array}{lll}
f(x) = & 2x^3 - 5x^2 - x + 10 & \underline{(x - 2} \; (= x - r) \\
 & \underline{2x^3 - 4x^2} & 2x^2 - x - 3 (= Q(x)) \\
 & \quad - x^2 - x & \\
 & \quad \underline{- x^2 + 2x} & \\
 & \qquad - 3x + 10 & \\
 & \qquad \underline{- 3x + 6} & \\
 & \qquad\qquad 4 (= R) &
\end{array}
$$

$$f(r) = f(2) = 2 \cdot 2^3 - 5 \cdot 2^2 - 2 + 10 = 2 \cdot 8 - 5 \cdot 4 - 2 + 10$$
$$= 16 - 20 - 2 + 10 = 4 = R.$$

104. Factor theorem.

If r is a root of $f(x) = 0$, then $x - r$ is a factor of $f(x)$.

By the preceding section, the remainder obtained by dividing $f(x)$ by $x - r$ is $R = f(r)$. But $f(r) = 0$, since r is a root of $f(x) = 0$. Therefore,

$$f(x) = (x - r)Q(x) + R = (x - r)Q(x),$$

or $x - r$ is a factor of $f(x)$.

105. Converse of the factor theorem.

Conversely, *if $x - r$ is a factor of $f(x)$, then r is a root of $f(x) = 0$.*

Since $x - r$ is a factor of $f(x)$, we can write

$$f(x) = (x - r)Q(x).$$

Therefore,

$$f(r) = (r - r)Q(r) = 0 \cdot Q(r) = 0.$$

106. Synthetic division.

The division of a polynomial by a binomial of the type $x - r$ can be effected simply and quickly by a process called **synthetic division,** which we shall illustrate by means of the example $2x^3 - 3x^2 - 13x + 5 \div x - 3$. By the ordinary process of long division we get

$$\begin{array}{rrrrl}
2x^3 & -\ 3x^2 & -\ 13x & +\ \ 5 & \underline{(x - 3} \\
\underline{2x^3} & \underline{-\ 6x^2} & & & 2x^2 + 3x - 4 \\
 & 3x^2 & -\ 13x & & \\
 & \underline{3x^2} & \underline{-\ \ \ 9x} & & \\
 & & -\ \ \ 4x & +\ \ 5 & \\
 & & \underline{-\ \ \ 4x} & \underline{+\ 12} & \\
 & & & -\ \ 7 &
\end{array}$$

Much of this is quite superfluous. In the first place it is unnecessary to write the first term in each line to be subtracted. Omitting these terms, and not bringing down the terms of the divisor, we have

$$\begin{array}{rrrrl}
2x^3 & -\ 3x^2 & -\ 13x & +\ \ 5 & \underline{(x - 3} \\
\underline{} & \underline{-\ 6x^2} & & & 2x^2 + 3x - 4 \\
 & 3x^2 & & & \\
 & & \underline{-\ \ 9x} & & \\
 & & -\ \ 4x & & \\
 & & & \underline{+\ 12} & \\
 & & & -\ \ 7 &
\end{array}$$

Next we write only the coefficients, and push these up into the more compact form shown below. The quotient has been omitted since the coefficients of the terms of the quotient, as well as the remainder, appear in the third line (provided we bring the first coefficient, 2, down into this line).

$$\begin{array}{rrrr|l} 2 & -3 & -13 & +5 & 1-3 \\ & -6 & -9 & +12 & \\ \hline 2 & +3 & -4 & -7 & \end{array}$$

Finally, to replace subtraction by addition, we replace (in the divisor) -3 by $+3$ and omit the 1:

$$\begin{array}{rrrr|l} 2 & -3 & -13 & +5 & 3 \\ & +6 & +9 & -12 & \\ \hline 2 & +3 & -4 & (-7) & \end{array}$$

Quotient $= 2x^2 + 3x - 4$, remainder $= -7$.

RULE. *To divide $f(x)$ by $x - r$, arrange $f(x)$ according to descending powers of x, being sure to supply each missing power with a zero as its coefficient. The work can be conveniently arranged in three lines.*

In the first line write the coefficients of $f(x)$ in order, thus: $a_0\ a_1\ a_2 \cdots a_n$. (Zero must be written for any missing power.)

Write a_0 in the first place in the third line. Multiply a_0 by r, place the product in the second line under a_1 and add, placing the sum in the third line. Multiply this by r, place the product in the second line under a_2 and add. Continue this process as far as possible.

The last sum in the third line will be the remainder, and the preceding numbers, reading from left to right, will be the coefficients of the powers of x in the quotient, arranged according to descending powers of x.

Example.

Divide $3x^4 - 5x^2 + x - 20$ by $x + 2$.

SOLUTION.

$$\begin{array}{l} 3 + 0 - \ 5 + \ 1 - 20 \quad (\underline{-2} \\ \quad - 6 + 12 - 14 + 26 \\ \hline 3 - 6 + \ 7 - 13 \ (+6) \end{array}$$

Quotient $= 3x^3 - 6x^2 + 7x - 13$, remainder $= 6$.

EXERCISES XIII. A

Use synthetic division in the following exercises:

1. $(x^3 + 4x^2 - 2x - 5) \div (x - 2)$.
2. $(x^3 - 5x^2 + 7x + 4) \div (x - 3)$.
3. $(x^3 + 3x^2 - 4x + 7) \div (x - 1)$.
4. $(x^3 + 6x^2 + 6x + 2) \div (x + 2)$.
5. $(x^3 + x^2 - 2x + 12) \div (x + 3)$.
6. $(4x^3 - 7x^2 + 15x - 17) \div (x + 1)$.
7. $(3x^3 - 32x - 60) \div (x - 4)$.
8. $(2x^3 + 7x^2 + 50) \div (x + 5)$.
9. $(x^4 + 2x^3 - 3x^2 + 4x - 5) \div (x - 1)$.
10. $(2x^4 + 5x^3 - 37x + 36) \div (x + 4)$.
11. $(4x^4 + 3x^2 - 1) \div (x - \frac{1}{2})$.
12. $(x^5 + x^4 - 2x^2 + 150) \div (x + 3)$.
13. $(3x^4 - 7x^3 + x - 2) \div (x + \frac{2}{3})$.
14. $(2x^5 - 6x^4 + 5x^3 - 15x^2 + 4x - 12) \div (x - 3)$.
15. $(3x^3 - 0.2x^2 + 0.30x - 0.023) \div (x - 0.3)$.
16. $(2x^3 + 1.7x^2 - 0.45x - 0.112) \div (x + 0.2)$.
17. $(x^3 + 3ax^2 - 6a^2x + 3a^3) \div (x - a)$.
18. $(2x^3 - 4x^2y - 7xy^2 + y^3) \div (x + 3y)$.
19. $(3x^4 - 4x^2 - 4) \div (x + \sqrt{2})$.
20. $(3x^4 - 4x^2 - 4) \div (x - i)$.
21. $(2x^4 + 3x^2 - 20) \div (x + 2i)$.
22. $(2x^3 + 3x^2 - 10x - 15) \div (x - \sqrt{5})$.

Use synthetic division and the remainder theorem in the following exercises:

23. Given $f(x) = x^3 - 3x^2 + 5x + 7$; find $f(2), f(-3)$.
24. Given $f(x) = 3x^4 - 4x^3 - 2x^2 - 5x + 8$; find $f(1), f(-2), f(3)$.
25. Given $f(x) = x^4 - 38x^2 + 5x + 42$; find $f(6), f(-6)$.

26. Show that $x - 5$ is a factor of $2x^3 - 3x^2 - 39x + 20$.
27. Show that $(x - 3)^2$ is a factor of $3x^4 - 16x^3 + 11x^2 + 42x - 36$.

107. Location of the real zeros of a polynomial.

Synthetic division is useful in graphing a polynomial, and consequently in locating its *real* zeros. For example, consider the polynomial

$$f(x) = 2x^3 - 7x^2 - 10x + 20.$$

The remainder theorem tells us that the remainder obtained when dividing by $x - r$ is $f(r)$. By synthetic division we can then construct the following table:

x	-3	-2	-1	0	1	2	3	4	5
$f(x)$	-67	-4	21	20	5	-12	-19	-4	45

If the values shown in this table are plotted we have a graph of the polynomial (Fig. 28). We note that it crosses the x-axis between -2 and -1, between 1 and 2, and between 4 and 5. Since $f(x) = 0$ wherever the graph crosses the x-axis, we have located three roots of the equation $f(x) = 0$.

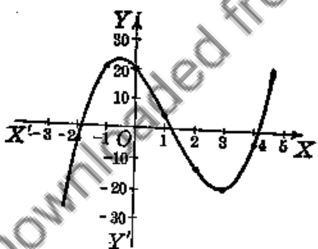


FIG. 28

Here we have made use of the following general principle, which applies to any polynomial $f(x)$:

PRINCIPLE. *If $f(a)$ and $f(b)$ have opposite signs, then $f(x) = 0$ for at least one value of x between a and b. That is, there is at least one real root of $f(x) = 0$ between a and b.*

This principle depends upon the fact that a polynomial is a **continuous** function and that its graph is a continuous curve, which means essentially that the graph is not composed of disconnected parts. If $f(a)$ and $f(b)$ have opposite signs, the points A and B on the graph, corresponding to $x = a$ and $x = b$ respectively, will be on opposite sides of the x-axis. Consequently the curve must cross the axis at least once, and in any case an odd number of times, between A and B. But to each crossing there corresponds a root of $f(x) = 0$.

As a consequence of this fundamental principle we have the following useful propositions:

If an integral rational equation of odd degree with real coefficients has the coefficient of its term of highest degree positive, the equation has at least one real root whose sign is opposite to that of the constant term.

Let the equation be

$$f(x) = a_0x^n + \cdots + a_n = 0, \qquad a_0 > 0. \qquad (1)$$

Since n is odd, $f(x)$ will be positive for large positive values of x and negative for large negative values.* It will have the value a_n for $x = 0$. Symbolically,

$$f(-\infty) = -\infty, \qquad f(0) = a_n, \qquad f(+\infty) = +\infty.$$

If a_n is positive there will be a root between $-\infty$ and 0, that is, a negative root. Similarly, if a_n is negative, there will be a positive root.

If an integral rational equation of even degree with real coefficients has the coefficient of its highest power positive, and has its constant term negative, the equation has at least one positive and at least one negative root.

If, in (1), n is even and $a_n < 0$, then $f(x)$ will be positive for large positive and large negative values of x, but will be negative when $x = 0$. It follows that there will be at least one positive and at least one negative root.

* This is because the sign of $f(x)$ is determined, for sufficiently large numerical values of x, by the sign of the term of highest degree. For example, $x^3 - 80x^2 - 50x - 1000$ will certainly be positive if x is 100 or more.

108. Upper and lower bounds for roots.

If r is positive (or zero), and upon dividing a polynomial $f(x)$ by $x - r$ (synthetic division) we find that all numbers in the third line are positive or zero, then r is an upper bound to the roots of the equation $f(x) = 0$. That is, no root can be greater than r.

For any number larger than r will make the numbers in the third line still larger, and consequently the remainder cannot be zero. Hence no such larger number can be a root.

Example 1.

Find an upper bound to the roots of

$$f(x) = 2x^3 - 7x^2 - 10x + 20 = 0.$$

SOLUTION. We have already found, in the preceding section, that there is a root between 4 and 5. Let us divide the polynomial by $x - 5$:

$$\begin{array}{rrrrl} 2 & -\ 7 & -\ 10 & +\ 20 & \underline{(5} \\ & +\ 10 & +\ 15 & +\ 25 & \\ \hline 2 & +\ 3 & +\ 5 & +\ 45 & \end{array}$$

All the numbers in the third line are positive: hence 5 is an upper bound to the roots.

To find a lower bound to the roots, use the equation $f(-x) = 0$, whose roots will be negatives of those of $f(x) = 0$. (That is, if $f(x) = 0$ has a root $x = -5$, then $f(-x) = 0$ will have a root $x = 5$.) *If r (a positive number) is an upper bound to the roots of $f(-x) = 0$, then $-r$ is a lower bound to the roots of $f(x) = 0$.*

The following rule, which is not difficult to demonstrate, may also be used in determining a lower bound to the roots of an integral rational equation, $f(x) = 0$: *If r is negative, and upon dividing $f(x)$ by $x - r$ (synthetic division) we find* *that the numbers in the third line alternate in sign,* then r is a lower bound to the roots of* $f(x) = 0$.

Example 2.

Find a lower bound to the roots of

$$f(x) = 2x^3 - 7x^2 - 10x + 20 = 0.$$

SOLUTION.

$$\begin{aligned} f(-x) &= 2(-x)^3 - 7(-x)^2 - 10(-x) + 20 \\ &= -2x^3 - 7x^2 + 10x + 20. \end{aligned}$$

Note that replacing x by −x changes the signs of all terms of odd degree but does not affect terms of even degree.

Set $f(-x)$ equal to zero and change signs:

$$2x^3 + 7x^2 - 10x - 20 = 0.$$

We have found in section 107 that $f(x)$ has a zero between -1 and -2; consequently $f(-x)$ will have a zero between 1 and 2. Let us show that 2 is an upper bound to the roots of $f(-x) = 0$.

$$\begin{array}{rrrrl} 2 & +\ 7 & -\ 10 & -\ 20 & \quad (\underline{2} \\ & +\ 4 & +\ 22 & +\ 24 & \\ \hline 2 & +\ 11 & +\ 12 & +\ 4 & \end{array}$$

All the numbers in the third line are positive; 2 is an upper bound to the roots of $f(-x) = 0$, and -2 is a lower bound to the roots of $f(x) = 0$.

EXERCISES XIII. B

Draw graphs of the following polynomials. Give exact values of their integral zeros and locate their other real zeros between consecutive integers.

1. $x^3 - 5x^2 + 2x + 8$. 2. $2x^3 - 15x^2 + 33x - 20$.

* A 0 may be counted as either + or −.

3. $3x^3 - 22x^2 + 29x + 30.$
4. $x^3 - 7x^2 + 36.$
5. $2x^3 + 3x^2 - 20x - 21.$
6. $2x^4 + 9x^3 - 8x^2 - 36x.$
7. $x^3 - 7x + 2.$
8. $x^3 - 7x^2 + 2.$
9. $x^3 - 4x^2 - 5x + 14.$
10. $4x^3 - 4x^2 - 29x + 15.$
11. $x^3 + 2x^2 - 3x + 10.$
12. $2x^3 + x^2 - x - 7.$
13. $3x^4 - 7x^3 + 7x^2 + 5x - 12.$
14. $2x^4 + x^3 - 7x^2 - 12x - 20.$
15. $x^4 - 2x^2 + 2.$
16. $x^4 - 2x^3 + 3x^2 - 8x - 4.$

Find, by the method of section 108, upper and lower bounds (nearest whole number above or below, respectively) for the roots of the following equations:

17. $x^3 + 4x^2 - 2x - 5 = 0.$
18. $x^3 - x^2 + 3x + 7 = 0.$
19. $3x^3 - 2x^2 - 6x - 5 = 0.$
20. $2x^3 + 7x^2 - 3x - 8 = 0.$
21. $x^3 - 12x + 4 = 0.$
22. $x^3 - 12x^2 + 4 = 0.$
23. $x^4 - 4x^3 - 8x^2 - 16x + 32 = 0.$
24. $x^4 + x^3 - 7x^2 - 9x - 100 = 0.$

109. Number of roots.

We assume the following theorem, which is called the **fundamental theorem of algebra**: *Every integral rational equation has at least one root.* (For proof see Louis Weisner, *Introduction to the Theory of Equations*, page 145, or L. E. Dickson, *New First Course in the Theory of Equations*, Appendix.)

Then we can prove:

THEOREM. *Every integral rational equation of degree n,*

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n = 0, \; a_0 \neq 0, \qquad (1)$$

has exactly n roots.

By the fundamental theorem, $f(x) = 0$ has at least one root. Let this root be r_1. Then, by the factor theorem, $x - r_1$ is a factor of $f(x)$ and we can write

$$f(x) = (x - r_1)Q_1(x),$$

where $Q_1(x)$ is a polynomial of degree $n - 1$, having a_0x^{n-1} as its term of highest degree.

Applying the fundamental theorem again, we find that $Q_1(x)$ has at least one zero, say r_2, and $x - r_2$ is a factor of $Q_1(x)$. Thus,

$$Q_1(x) = (x - r_2)Q_2(x),$$

and

$$f(x) = (x - r_1)(x - r_2)Q_2(x),$$

where $Q_2(x)$ is a polynomial of degree $n - 2$, whose term of highest degree is a_0x^{n-2}.

Continuing this process n times we get

$$f(x) = (x - r_1)(x - r_2) \cdots (x - r_n)Q_n(x),$$

where $Q_n(x)$ is a polynomial of degree $n - n = 0$, whose term of highest degree is $a_0x^0 = a_0$. That is, Q_n is the constant a_0. Thus,

$$f(x) = a_0(x - r_1)(x - r_2) \cdots (x - r_n), \qquad (2)$$

and $f(x) = 0$ has the n roots $r_1, r_2, \cdots, r_n$.

The equation can have no other roots, for no other number, such as r, when substituted for x will make any of the factors on the right side of (2) equal to zero.

It should be noted that the numbers $r_1, r_2, \cdots, r_n$ do not have to be real. Also they are not necessarily all distinct. For example, we might have $x - r$ occurring m times as a factor of $f(x)$, in which case r is called a **multiple root** of $f(x) = 0$ **of order** m, or a **root of multiplicity** m. A root of multiplicity 2 is called a **double root,** a root of multiplicity 3 is a **triple root,** and so on. A root corresponding to a linear factor which occurs only once is called a **simple root.**

As a corollary of the foregoing theorem we have the following:

If two polynomials in the same variable, each of degree not greater than n, are equal in value for more than n distinct values of the variable, then the polynomials are identical.

Let the polynomials be

$$a_0x^n + a_1x^{n-1} + \cdots + a_n, \qquad b_0x^n + b_1x^{n-1} + \cdots + b_n.$$

If they are equal for more than n values of x, their difference is zero for more than n values of x. This difference may be written in the form:

$$(a_0 - b_0)x^n + (a_1 - b_1)x^{n-1} + \cdots + (a_n - b_n) = 0. \quad (3)$$

But if one or more of the coefficients $(a_0 - b_0)$, $(a_1 - b_1)$, $\cdots$, $(a_n - b_n)$ in (3) were different from zero, we should have an equation of degree n, or less, with more than n distinct roots. This contradicts the foregoing theorem and is therefore impossible. Thus, each of the coefficients in (3) must be zero, and we have

$$a_0 = b_0, \qquad a_1 = b_1, \qquad \cdots, \qquad a_n = b_n.$$

In other words, the two polynomials are identical; their values are equal for all values of x.

110. Imaginary roots.

If an imaginary number $a + bi$ is a root of an integral rational equation $f(x) = 0$ with real coefficients, the conjugate imaginary number $a - bi$ is also a root.

Since $a + bi$ is a root of $f(x) = 0$, then by the factor theorem, $x - (a + bi)$ is a factor of $f(x)$. The method of proof will be to show that $x - (a - bi)$ is also a factor of $f(x)$ by showing that the product of these two linear factors is a factor of $f(x)$. Denoting this product by $P(x)$, we have

$$\begin{aligned} P(x) &= [x - (a + bi)][x - (a - bi)] \\ &= x^2 - 2ax + a^2 + b^2. \end{aligned}$$

Divide $f(x)$ by $P(x)$ until the remainder is of degree not higher than the first (that is, until it is linear or constant).

Denoting the quotient by $Q(x)$ and the remainder by $Rx + S$, we have the identity

$$f(x) \equiv P(x) \cdot Q(x) + Rx + S, \tag{1}$$

in which R and S are real because $P(x)$ has real coefficients.

In the above identity set $x = a + bi$. By hypothesis, $f(a + bi) = 0$. Since $x - (a + bi)$ is a factor of $P(x)$, it follows by the converse of the factor theorem that $P(a + bi) = 0$. Therefore from (1) we obtain the relation

$$0 = 0 \cdot Q(a + bi) + R \cdot (a + bi) + S, \tag{2}$$

or

$$Ra + S + Rbi = 0. \tag{3}$$

Since an imaginary number cannot be zero unless both its real part and its imaginary part are zero (see section 92), it follows from (3) that

$$Ra + S = 0, \qquad Rb = 0. \tag{4}$$

But, since $a + bi$ is imaginary, $b \neq 0$. From the second equation of (4) it follows that $R = 0$, and then, from the first equation of (4), that $S = 0$. That is, the remainder $Rx + S$ in (1) is zero, and $f(x) = P(x) \cdot Q(x)$. In other words, $P(x)$ is a factor of $f(x)$.

But $x - (a - bi)$ is a factor of $P(x)$, and consequently of $f(x)$. Therefore, $a - bi$ is a root of $f(x) = 0$.

111. Quadratic surd roots.

A radical like $\sqrt{3}$ or $\sqrt[3]{5}$, in which the number under the radical sign is rational but the radical itself is irrational, is called a **surd.** A surd is called **quadratic, cubic,** and so on, according as its index is two, three, and so on.

A theorem for quadratic surds, similar to the theorem in the foregoing section, is as follows:

If a quadratic surd $a + \sqrt{b}$ (a rational) is a root of an integral rational equation with rational coefficients, the conjugate surd $a - \sqrt{b}$ is also a root.

The proof follows a line of reasoning similar to that used for the theorem on conjugate imaginary roots, and will not be given here.

EXERCISES XIII. C

1. Solve the equation $x^4 - x^3 - 16x^2 + 59x + 13 = 0$, given that one of its roots is $3 + 2i$.

   SOLUTION. Since $3 + 2i$ is a root, $3 - 2i$ is a root. Therefore the expression

   $$[x - (3 + 2i)]\,[x - (3 - 2i)] = x^2 - 6x + 13$$

   is a factor of the left side of the given equation. Dividing by this factor, we get $x^2 + 5x + 1 = 0$, a quadratic equation whose roots are $\frac{1}{2}(-5 \pm \sqrt{21})$. The complete solution is

   $$x = 3 \pm 2i, \quad \tfrac{1}{2}(-5 \pm \sqrt{21}).$$

2. Solve the equation $x^4 - 7x^3 + 11x^2 - 5x - 2 = 0$, given that $2 + \sqrt{3}$ is a root.
3. Solve the equation $2x^4 + 9x^3 + 17x^2 - 11x - 45 = 0$, given that $-2 + i\sqrt{5}$ is a root.
4. Solve the equation $2x^4 - 8x^3 + 31x^2 - 51x + 78 = 0$, given that $\frac{1}{2}(1 + 5i)$ is a root.
5. Solve the equation $x^6 - 4x^5 - 4x^4 - 8x^3 - 11x^2 - 4x - 6 = 0$, given that i is a double root.
6. Solve the equation $x^6 - 4x^5 + 6x^4 - 12x^2 + 16x - 8 = 0$, given that $1 - i$ is a double root.

112. Graph of a factored polynomial.

When a polynomial, or integral rational function, $f(x)$, is expressed in factored form, its graph can be quickly sketched. At each *real* root of $f(x) = 0$ the graph meets the x-axis. At each simple root the curve crosses the axis at an angle, at each multiple root of even order it is tangent to the axis and does not cross it, at each multiple root of odd order it is tangent to the axis and crosses it. A graph of the polynomial

$$y = a_0(x - r_1)(x - r_2)^3(x - r_3)^2 \qquad (a_0 > 0),$$

in which the r's are all real, would appear as in Fig. 29.

If the polynomial contains any imaginary factors they occur in conjugate pairs (section 110). These imaginary factors do not affect the manner in which the curve meets the x-axis, but do affect the shape of the curve.

FIG. 29

Proofs of these statements are not difficult but will be omitted here.

EXERCISES XIII. D

Sketch the graphs of the following polynomials:

1. $(x + 2)(x - 3)(x - 5)$.
2. $(x + 3)(x - 1)^2(x + 4)$.
3. $(x + 5)^2(x + 3)(x - 1)$.
4. $2x(x + 2)(x - 2)$.
5. $\frac{1}{2}(x + 1)^2(x - 3)^3$.
6. $-(x + 2)(x - 1)^2(x - 4)^3$.
7. $(x - 1)(x - 4)^4$.
8. $\frac{1}{5}(2x - 3)^3(x + 3)^2$.
9. $-(x + 3)(x - 1)^3(x - 3)^3$.
10. $(x^2 - 4)(x - 4)^2$.
11. $(x + 6)(x + 3)(x - 1)(x - 3)(x - 4)$.
12. $x^2(x + 1)(x - 2)^3(x - 4)^2$.
13. $(3 - x)(1 - x)^2(4 + x)$.
14. $(x + 1)(x - 1)^2(x - 2)^3(x - 3)^4$.

113. Descartes' rule of signs.

Two consecutive terms of a polynomial in x, with real coefficients, when the terms are arranged according to descending powers of x, are said to have a **variation** in sign when one is plus and the other minus. Thus, the polynomial

$$2x^5 - 3x^4 - 4x^3 + 3x - 7$$

has three variations in sign. (Note that some powers of x may be missing.) **Descartes' rule of signs** states:

An integral rational equation $f(x) = 0$, with real coefficients, has as many positive roots as it has variations of signs, or fewer by an even number. (A root of multiplicity m is counted as m roots.)

Suppose that a polynomial $P(x)$ is represented by the following scheme, in which the dots after a sign indicate that there may be more signs of the same kind, but that no change in sign can occur before the next explicitly written sign:

$$P(x) = + \cdots - \cdots + \cdots.$$

Multiply $P(x)$ by $x - r (r > 0)$. Schematically the multiplication may be represented by

$$\begin{aligned} xP(x) &= + \cdots\cdots - \cdots\cdots + \cdots\cdots \\ -rP(x) &= \quad - \cdots - + \cdots + - \cdots - \\ (x - r)P(x) &= + \pm \cdots - \pm \cdots + \pm \cdots - \end{aligned}$$

The ambiguous sign $\pm$ indicates that the sign of the corresponding term is undetermined—we do not know whether it is plus or minus. But no matter what signs these doubtful terms may have, there is at least one more variation in sign in the product $(x - r)P(x)$ than in $P(x)$, since there is an additional variation at the end.

The product may contain more than one additional variation; for successive terms of like signs, for example, $+ + +$ or $- - -$, in the original polynomial, which are replaced in the product by ambiguities may actually be replaced by $+ - +$ or $- + -$ respectively. But such changes always increase the number of variations by an even number.

Hence the number of variations in $(x - r)P(x)$ exceeds that in $P(x)$ by 1, or by 1 plus an even number, that is, by an odd number.

Now suppose that the product of all the factors corresponding to negative and complex roots of $f(x)$ has been formed into a polynomial $P(x)$. Since $P(x) = 0$ has no positive roots its first and last terms must have like signs. (Otherwise it would be negative for $x = 0$ and positive for large values of x, or vice versa, and would have a positive root.) Therefore, $P(x)$ must have an even number of variations in sign, say $2k$.

If a positive root is introduced by multiplying $P(x)$ by $x - r_1$ $(r_1 > 0)$, the number of variations is increased by an odd number, say $2k_1 + 1$. Suppose that we have introduced m positive roots, $r_1, \cdots, r_m$. We have increased the number of variations to

$$2k + 2k_1 + 1 + \cdots + 2k_m + 1$$
$$= 2(k + k_1 + \cdots + k_m) + m.$$

That is, the number of positive roots, m, is the number of variations decreased by $2(k + k_1 + \cdots + k_m)$, which is an even number (zero if all the k's are zero). But this is Descartes' rule.

Information concerning the number of negative roots of an equation may be obtained by applying Descartes' rule to $f(-x) = 0$.

Example 1.

Discuss the nature of the roots of

$$f(x) = x^3 + 3x - 5 = 0.$$

SOLUTION. 1 variation; therefore 1 positive root.

$$f(-x) = -x^3 - 3x - 5 = 0.$$

No variation; therefore no negative roots of the original equation.

The equation is of third degree and must have three roots. Thus, two roots must be imaginary.

Example 2.

Discuss the nature of the roots of

$$f(x) = 2x^5 - 3x^4 - 4x^3 + 3x - 7 = 0.$$

SOLUTION. 3 variations, therefore 3 or 1 positive roots.

$$f(-x) = -2x^5 - 3x^4 + 4x^3 - 3x - 7 = 0.$$

2 variations; therefore 2 or 0 negative roots.

The various possibilities for the kinds of roots are shown below.

Positive	3	3	1	1
Negative	2	0	2	0
Imaginary	0	2	2	4
Total	5	5	5	5

EXERCISES XIII. E

Give all of the information obtainable from Descartes' rule of signs about the roots of the following equations:

1. $x^3 + 7x - 5 = 0.$
2. $x^3 + 7x^2 - 5 = 0.$
3. $2x^3 + 3x^2 - 4x - 6 = 0.$
4. $5x^3 - x^2 + 10x - 8 = 0.$
5. $3x^3 - 4x^2 - 6x + 7 = 0.$
6. $x^3 + 6x^2 + 5x - 2 = 0.$
7. $7x^3 + 2x^2 - 5x + 4 = 0.$
8. $2x^3 - 3x^2 + 10x + 1 = 0.$
9. $x^3 + 12x^2 + 6x - 3 = 0.$
10. $4x^3 + 3x^2 + 2x + 1 = 0.$
11. $3x^3 - 5x^2 + 7x - 9 = 0.$
12. $x^3 - 2x^2 - 3x + 4 = 0.$
13. $9x^3 - 3x^2 + x + 7 = 0.$
14. $6x^3 + 11x^2 - 7x - 9 = 0.$
15. $\frac{1}{2}x^3 + \frac{1}{3}x^2 + \frac{1}{4}x - \frac{1}{5} = 0.$
16. $\frac{1}{5}x^3 - \frac{1}{3}x^2 - \frac{1}{4}x - \frac{1}{5} = 0.$
17. $x^3 + 3x^2 + 5x + 7 = 0.$
18. $x^3 + 9x^2 - 4x + 6 = 0.$
19. $x^4 + x^3 + x + 1 = 0.$
20. $x^4 + x^2 + x + 1 = 0.$
21. $4x^4 - 3x^3 + 2x^2 + x + 5 = 0.$
22. $2x^4 + 3x^3 - 5x^2 + x + 7 = 0.$

23. $3x^4 + 5x^3 + 7x^2 - 9x + 10 = 0.$
24. $x^4 - 2x^3 - 3x^2 + 4x - 5 = 0.$
25. $5x^4 - 3x^3 + 2x^2 - 10x - 12 = 0.$
26. $6x^4 - 7x^3 + 3x^2 - 11 = 0.$
27. $3x^4 - 9x^2 - 2x + 5 = 0.$
28. $4x^4 + 3x^3 + 2 = 0.$ **29.** $4x^4 + 3x^2 + 2 = 0.$
30. $x^4 - 20 = 0.$ **31.** $x^5 + 20 = 0.$
32. $x^6 - 1 = 0.$ **33.** $x^6 - x^3 + 1 = 0.$

114. Rational roots.

A **rational number** is one which may be exactly expressed as the ratio of two whole numbers, for example, $\frac{2}{3}$, $-\frac{17}{6}$, 5, 0. Any number which cannot be so expressed is an **irrational number,** for example, $\sqrt{2}$, π.

THEOREM. *If a rational number b/c, a fraction in its lowest terms or an integer,* is a root of the integral rational equation*

$$a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0 \qquad (1)$$

with integral coefficients, then b is a factor† of a_n and c is a factor of a_0.

To prove this, substitute b/c in (1):

$$a_0\left(\frac{b}{c}\right)^n + a_1\left(\frac{b}{c}\right)^{n-1} + \cdots + a_{n-1}\frac{b}{c} + a_n = 0. \qquad (2)$$

Multiply both sides of the equation by c^n:

$$a_0b^n + a_1b^{n-1}c + \cdots + a_{n-1}bc^{n-1} + a_nc^n = 0. \qquad (3)$$

Transpose the last term to the right side and factor b from the expression remaining on the left:

$$b(a_0b^{n-1} + a_1b^{n-2}c + \cdots + a_{n-1}c^{n-1}) = -a_nc^n. \qquad (4)$$

Since b is a factor of the left side it is a factor of the right

* If b/c is an integer, then c is to be taken as 1.
† When we say that b is a factor of a_n we mean here that the integer b is contained in the integer a_n a whole number of times.

side. It cannot be a factor of c, since b/c is in its lowest terms. It must therefore be a factor of a_n.

Similarly, by transposing the first term in (3), we get

$$c(a_1b^{n-1} + \cdots + a_{n-1}bc^{n-2} + a_nc^{n-1}) = -a_0b^n. \quad (5)$$

Arguing as before, we see that c must be a factor of the right side, and since it is not a factor of b it must be a factor of a_0.

COROLLARY. *Any rational root of the equation*

$$x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0, \quad (6)$$

in which the a's are integers, is an integer and is a factor of a_n.

Rational roots can be found by synthetic division.

Example.

Find the rational roots of

$$f(x) = 2x^4 + 9x^3 + 15x^2 + 13x + 6 = 0.$$

SOLUTION.

Possible numerators (factors of 6): 1, 2, 3, 6.
Possible denominators (factors of 2): 1, 2.
Possible rational roots: $\pm(1, 2, 3, 6, \frac{1}{2}, \frac{3}{2})$.

By Descartes' rule there are no positive roots of any kind, consequently we only need to try negative values. In looking for negative roots it is usually best to change to $f(-x) = 0$. Here,

$$f(-x) = 2x^4 - 9x^3 + 15x^2 - 13x + 6:$$

4 variations; therefore 4, 2, or 0 negative roots.

We test the possible roots in order, beginning with the smallest.*

$$\begin{array}{l} 2 - 9 + 15 - 13 + 6 \quad (\tfrac{1}{2} \\ \quad\ 1 - 4 \\ \hline 2 - 8 + 11 \end{array} \qquad \begin{array}{l} 2 - 9 + 15 - 13 + 6 \quad (1 \\ \quad\ 2 - 7 + 8 - 5 \\ \hline 2 - 7 + 8 - 5(+1) \end{array}$$

Note that it is unnecessary to complete the synthetic division for 1/2. For $1/2 \times 11$ is a fraction, and it would be impossible to come out with a zero remainder.

$$\begin{array}{l} 2 - 9 + 15 - 13 + 6 \quad (\tfrac{3}{2} \\ \quad\ 3 - 9 + 9 - 6 \\ \hline 2 - 6 + 6 - 4(+0) \end{array}$$

$\frac{3}{2}$ is a root of $f(-x) = 0$,
$-\frac{3}{2}$ is a root of $f(x) = 0$.

We now use the depressed equation, $2x^3 - 6x^2 + 6x - 4 = 0$, represented by the numbers $2 - 6 + 6 - 4$, which is the quotient obtained by dividing $f(-x) = 0$ by $x - \frac{3}{2}$. This can be simplified, by division by 2, to $x^3 - 3x^2 + 3x - 2 = 0$. It has the same roots as $f(-x) = 0$, with the exception of 3/2, which has been divided out.

For the depressed equation we have

Possible numerators: 1, 2.
Possible denominators: 1.
Possible rational roots: 1, 2.

We have already tried 1 in $f(-x)$. Try 2:

$$\begin{array}{l} 1 - 3 + 3 - 2 \quad (2 \\ \quad\ 2 - 2 + 2 \\ \hline 1 - 1 + 1(+0) \end{array}$$

2 is a root of the depressed equation,
-2 is a root of $f(x) = 0$.

The new depressed equation is $x^2 - x + 1 = 0$. Solving by the quadratic formula, we get

$$x = \frac{1 \pm \sqrt{1 - 4}}{2} = \frac{1 \pm i\sqrt{3}}{2}.$$

* By beginning with the numbers which are smallest numerically, we can sometimes find bounds for the roots, doing away with the necessity of testing any larger values.

The roots of $f(x) = 0$ are negatives of these. The original equation was of degree 4, and we have found all 4 roots; they are

$$-\frac{3}{2}, \quad -2, \quad \frac{-1 \pm i\sqrt{3}}{2}.$$

Note that even though a certain number has been found to be a root it may possibly be a root of the depressed equation. This will be the case if it is a multiple root of the original equation.

EXERCISES XIII. F

Find the rational roots of the following equations:

1. $x^3 - x^2 - 7x + 3 = 0.$ **2.** $x^3 - x + 6 = 0.$
3. $x^3 - 9x^2 + 17x + 12 = 0.$ **4.** $x^3 + 10x^2 + 11x - 40 = 0.$
5. $2x^3 + 9x^2 - 3x - 1 = 0.$ **6.** $3x^3 - 10x^2 - 11x - 2 = 0.$
7. $12x^3 + 4x^2 - 17x + 6 = 0.$
8. $8x^3 + 38x^2 - 13x - 15 = 0.$
9. $x^3 - \frac{3}{2}x^2 + \frac{1}{2}x - 3 = 0.$ **10.** $x^3 + \frac{7}{3}x^2 - \frac{7}{3}x - 1 = 0.$
11. $x^3 + 6x^2 - 36 = 0.$ **12.** $3x^3 - 11x^2 - 16x + 12 = 0.$
13. $6x^4 - 2x^3 + 5x^2 - 4x = 0.$
14. $x^4 - 7x^3 + 7x^2 - 3x - 18 = 0.$
15. $4x^4 + 12x^3 + x^2 - 24x - 18 = 0.$
16. $3x^4 - 7x^3 + 4x^2 - 12x + 16 = 0.$
17. $5x^4 + 13x^3 + 69x^2 - 25x - 2 = 0.$
18. $6x^4 - 25x^3 - 24x^2 + 121x + 42 = 0.$
19. $6x^4 + 19x^3 - 101x^2 - 254x - 120 = 0.$
20. $12x^4 + 23x^3 + 10x^2 - 4x - 5 = 0.$
21. In Fig. 30 the tangent TP is 9 inches in length. The diameter TR is extended 3 inches to the point Q. The tangent QS, extended, passes through P. Find the radius of the circle.

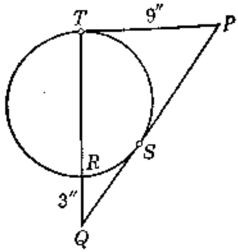


Fig. 30

115. Irrational roots.

Irrational roots of any type of equation $f(x) = 0$ can be found by a process of successive approximations, using the general principle stated in section 107, that if $f(a)$ and

$f(b)$ have opposite signs there is a root of $f(x) = 0$ between a and b. If the equation is an integral rational equation we may employ synthetic division to obtain values of $f(x)$, or we may obtain them by actual substitution. For other types of equation, such as*

$$\log x - x + 2 = 0,$$

we must use actual substitution.

Example.

Find an irrational root of $f(x) = x^3 + x - 5 = 0$.

SOLUTION. By Descartes' rule the equation has one positive root and no negative root, hence two imaginary roots.

By synthetic division, or by actual substitution, we find that $f(1) = -3$, $f(2) = 5$. Thus, the curve $y = f(x)$ crosses the x-axis between $x = 1$ and $x = 2$. (See Fig. 31.) A first approximation to the root is $x = 1$.

To get a better approximation we add h_1 to 1. To find h_1 we use similar triangles: †

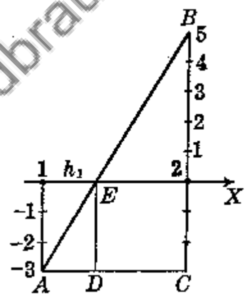


FIG. 31

$$\frac{AD}{AC} = \frac{DE}{CB}, \qquad \text{or} \qquad \frac{h_1}{1} = \frac{3}{8}.$$

$$h_1 = 0.4 \text{ approximately.}$$

A second approximation to the root is $x = 1.4$.

We now find $f(x)$ for x by tenths, using synthetic division or straight substitution. We shall use the latter, as it is quite general in its application, synthetic division being applicable only to polynomials. Using tables of cubes or a calculating machine, we find

* The symbol $\log x$ means the logarithm of x. (See Chapter XIV.)

† This assumes that the graph of the equation $y = f(x)$ from A to B is a straight line. Actually it is a curve, of which AB is a chord.

$$(1.4)^3 = 2.744 \quad (1.5)^3 = 3.375 \quad (1.6)^3 = 4.096$$

$(1.4)^3 =$	2.744	$(1.5)^3 =$	3.375	$(1.6)^3 =$	4.096
	1.4		1.5		1.6
	4.144		4.875		5.696
	−5		−5		−5
$f(1.4) =$	−0.856	$f(1.5) =$	−0.125	$f(1.6) =$	0.696

We see that the curve crosses the x-axis between 1.5 and 1.6. To

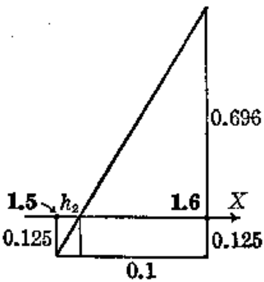


FIG. 32

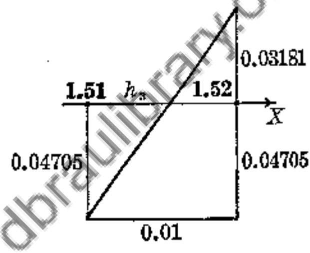


FIG. 33

get a better approximation to the root we add h_2 to 1.5. Again we find, by similar triangles,

$$\frac{h_2}{0.1} = \frac{0.125}{0.125 + 0.696} = \frac{0.125}{0.821}, \qquad h_2 = 0.015+.$$

To carry the result still further we proceed as before, taking x by hundredths.

$(1.51)^3 =$	3.442951	$(1.52)^3 =$	3.511808
	1.51		1.52
	4.952951		5.031808
	−5		−5
$f(1.51) =$	−0.047049	$f(1.52) =$	0.031808

$$\frac{h_3}{0.01} = \frac{0.047049}{0.047049 + 0.031808}$$

$$= \frac{0.047049}{0.078857}.$$

$$h_3 = 0.0060-.$$

Our next approximation to the root is

$$x = 1.51 + 0.0060 = 1.5160.$$

The process can be continued as far as desired, although the successive steps become more laborious.

EXERCISES XIII. G

Find, correct to three decimal places, the irrational roots of the following equations:

1. $x^3 + x - 4 = 0.$
2. $x^3 - x^2 - 5 = 0.$
3. $x^3 - 3x - 1 = 0.$
4. $x^3 + 9x + 6 = 0.$
5. $x^3 + 4x + 24 = 0.$
6. $x^3 - 3x + 6 = 0.$
7. $4x^3 - 9x - 6 = 0.$
8. $2x^3 - 9x^2 - 9 = 0.$
9. $x^3 - 6x + 2 = 0.$
10. $x^3 - 3x^2 - 2x + 3 = 0.$
11. $x^3 - 3x^2 + 4x - 5 = 0.$
12. $x^3 + 3x^2 + 4x + 3 = 0.$
13. $x^4 - 2x - 50 = 0.$
14. $x^4 - x^3 - 50 = 0.$
15. $x^3 - 5 = 0.$
16. $x^5 - 2 = 0.$
17. $x^4 - 3x^2 - 6x - 2 = 0.$
18. $x^5 - 40x - 100 = 0.$

116. Transformation to diminish the roots of an equation by a fixed amount.

In order to develop a more systematic but more technical method of finding irrational roots of integral rational equations, it is necessary to be able to transform an equation into a new equation whose roots are less, by a given amount, than those of the original equation.

Suppose that the original equation is

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0. \quad (1)$$

If we replace x by $x' + h$, we get a new equation in x', and if $x = r$ is a root of the original equation, then $x' = r - h$ will be a root of the transformed equation. That is, each root of the new equation will be less by an amount h than the corresponding root of the old equation.

Setting $x = x' + h$ in (1) we get

$$a_0(x' + h)^n + a_1(x' + h)^{n-1} + \cdots + a_{n-1}(x' + h) + a_n = 0, \quad (2)$$

or

$$a_0x'^n + A_1x'^{n-1} + \cdots + A_{n-1}x' + A_n = 0. \quad (3)$$

We can determine the A's by the artifice of changing back to (1) again by setting $x' = x - h$ in (3). This gives

$$a_0(x - h)^n + A_1(x - h)^{n-1} + \cdots + A_{n-1}(x - h) + A_n = 0, \quad (4)$$

which is merely $f(x)$ arranged in powers of $x - h$.

It will be noted that A_n is the remainder if $f(x)$ is divided by $x - h$, A_{n-1} is the remainder if the resulting quotient is divided by $x - h$, and so on.

Example.

Find an equation whose roots are less by 3 than those of $x^4 - 5x^3 + 20x - 16 = 0$.

SOLUTION.

$$\begin{array}{rrrrrl}
1 & -5 & +0 & +20 & -16 & \underline{(3} \\
 & 3 & -6 & -18 & +6 & \\
\hline
1 & -2 & -6 & +2 & \mathbf{-10} & \\
 & 3 & +3 & -9 & & \\
\hline
1 & +1 & -3 & \mathbf{-7} & & \\
 & 3 & +12 & & & \\
\hline
1 & +4 & \mathbf{+9} & & & \\
 & 3 & & & & \\
\hline
\mathbf{1} & \mathbf{+7} & & & &
\end{array}$$

The transformed equation is

$$x'^4 + 7x'^3 + 9x'^2 - 7x' - 10 = 0.$$

EXERCISE

Check the result by verifying that the roots of the original equation are -2, 1, 2, 4 and those of the transformed equation -5, -2, -1, 1.

EXERCISES XIII. H

Obtain equations whose roots are equal to the roots of the following equations diminished by the number indicated:

1. $x^3 - 3x^2 + 2x + 4 = 0$, 1.
2. $x^3 - 7x^2 - x + 3 = 0$, 2.
3. $x^3 + 6x^2 + 4x - 1 = 0$, -2.
4. $2x^3 - 8x^2 + 7x - 2 = 0$, 3.
5. $4x^3 + 12x^2 + 5x - 3 = 0$, -1.
6. $3x^3 - 36x^2 - 144x - 199 = 0$, 4.
7. $2x^3 - 5x^2 - 6x - 7 = 0$, 0.2.
8. $x^3 - 2x^2 + 3x - 5 = 0$, $\frac{2}{3}$.
9. $x^4 - 20x^3 + 50x^2 + 75x + 100 = 0$, 5.
10. $3x^4 + 12x^3 + 25x^2 + x + 4 = 0$, -1.

117. Horner's method.

This method of finding an irrational root of an integral rational equation is best explained by means of an example.

Example.

Find an irrational root of

$$f(x) = x^3 + x - 5 = 0. \tag{1}$$

By synthetic division (or by substitution) we find $f(1) = -3$, $f(2) = 5$. Thus, $f(x)$ has a root between 1 and 2. Transform (1) into an equation whose roots are less by 1 than those of (1), so that the new equation will have a root between 0 and 1:

$$\begin{array}{llll|l}
1 & +0 & +1 & -5 & \underline{1} \\
 & 1 & +1 & +2 & \\
\hline
1 & +1 & +2 & \mathbf{-3} & \\
 & 1 & +2 & & \\
\hline
1 & +2 & \mathbf{+4} & & \\
 & 1 & & & \\
\hline
\mathbf{1} & \mathbf{+3} & & &
\end{array}$$

The transformed equation is

$$f_1(x_1) = x_1^3 + 3x_1^2 + 4x_1 - 3 = 0. \tag{2}$$

When x_1 is small, x_1^3 is small and we can get a fairly good approximation to the root of (2) by solving the quadratic equation $3x_1^2 + 4x_1 - 3 = 0$, getting $x_1 = 0.5+$. (The positive root is the only one considered.) By synthetic division, we find $f_1(0.6) = 0.696$, $f_1(0.5) = -0.125$. The fundamental principle which must be kept in mind is that if $f(a)$ and $f(b)$ have opposite signs, there is a root of $f(x) = 0$ between a and b.

Transform (2) into an equation whose roots are less by 0.5:

$$
\begin{array}{llll|l}
1 & +3.0 & +4.00 & -3.000 & (0.5 \\
  & 0.5 & +1.75 & +2.875 & \\
\hline
1 & +3.5 & +5.75 & \mathbf{-0.125} & \\
  & 0.5 & +2.00 & & \\
\hline
1 & +4.0 & \mathbf{+7.75} & & \\
  & 0.5 & & & \\
\hline
\mathbf{1} & \mathbf{+4.5} & & &
\end{array}
$$

The transformed equation is

$$f_2(x_2) = x_2^3 + 4.5x_2^2 + 7.75x_2 - 0.125 = 0. \tag{3}$$

To get an approximation to the root of (3), neglect $x_2^3 + 4.5x_2^2$ and set $7.75x_2 - 0.125 = 0$, getting $x_2 = 0.016$. We find $f(0.02) = 0.031808$, $f(0.01) = -0.047049$.

Transform (3) into an equation whose roots are less by 0.01:

$$
\begin{array}{llll|l}
1 & +4.50 & +7.7500 & -0.125000 & (0.01 \\
  & 0.01 & +0.0451 & +0.077951 & \\
\hline
1 & +4.51 & +7.7951 & \mathbf{-0.047049} & \\
  & 0.01 & +0.0452 & & \\
\hline
1 & +4.52 & \mathbf{+7.8403} & & \\
  & 0.01 & & & \\
\hline
\mathbf{1} & \mathbf{+4.53} & & &
\end{array}
$$

The next transformed equation is

$$f_3(x_3) = x_3^3 + 4.53x_3^2 + 7.8403x_3 - 0.047049 = 0. \tag{4}$$

Setting $7.8403x_3 - 0.047049 = 0$, we get $x_3 = 0.0060$.

The root of $f(x) = 0$ is

$$x = 1 + 0.5 + 0.01 + 0.0060 = 1.5160.$$

This is certainly correct to two decimal places and very probably to three. The work can be continued further if desired.

The solution may be compactly exhibited as follows:

$$
\begin{array}{llllll}
1 + 0 & + 1 & - 5 & (1 & & \\
\quad 1 & + 1 & + 2 & & & \\
\hline
1 + 1 & + 2 | & - 3 & & & \\
\quad 1 & + 2 | & & & & \\
\hline
1 + 2 | & + 4 & & & & \\
\quad 1 | & & & & & \\
\hline
1 + 3.0 & + 4.00 & - 3.000 & (0.5 & & 3x_1^2 + 4x_1 - 3 = 0, \\
\quad 0.5 & + 1.75 & + 2.875 & & & x_1 = 0.5. \\
\hline
1 + 3.5 & + 5.75 | & - 0.125 & & & \\
\quad 0.5 & + 2.00 | & & & & \\
\hline
1 + 4.0 | & + 7.75 & & & & \\
\quad 0.5 | & & & & & \\
\hline
1 + 4.50 & + 7.7500 & - 0.125000 & (0.01 & & 7.75x_2 - 0.125 = 0, \\
\quad 0.01 & + 0.0451 & + 0.077951 & & & x_2 = 0.016. \\
\hline
1 + 4.51 & + 7.7951 | & - 0.047049 & & & \\
\quad 0.01 & + 0.0452 | & & & & \\
\hline
1 + 4.52 | & + 7.8403 & & & & 7.8403x_3 - 0.047049 = 0, \\
\quad 0.01 | & & & & & x_3 = 0.0060. \\
\hline
1 + 4.53 & & & & & x = 1.5160.
\end{array}
$$

To find a negative root of $f(x) = 0$ by Horner's method, find the corresponding positive root of $f(-x) = 0$ and prefix a minus sign.

118. Suggestions for finding the real roots of a numerical equation.

Obtain as much information as possible about the nature of the roots from Descartes' rule of signs.

Remove all rational roots, depressing the equation each time that such a root is removed.

Always take advantage of any information revealed about upper and lower bounds for the roots while looking for either rational or irrational roots.

After all rational roots have been removed find the irrational roots of the depressed equation by the method of section 115 or by Horner's method.

If it is possible to reduce the equation to a quadratic equation, the solution can be completed by solving this quadratic.

Negative roots are usually best found by solving $f(-x) = 0$.

EXERCISES XIII. I

Find, correct to three decimal places, all of the real roots of the following equations:

1. $x^3 + 2x - 5 = 0$.
2. $x^3 + 2x + 5 = 0$.
3. $x^3 + 6x^2 + 13x + 14 = 0$.
4. $x^3 - 3x^2 + 4x - 5 = 0$.
5. $x^3 - 5x - 3 = 0$.
6. $x^3 - 9x^2 + 24x - 17 = 0$.
7. $2x^3 + 6x^2 + 12x + 11 = 0$.
8. $3x^3 - 15x^2 + 21x - 11 = 0$.
9. $x^4 - 7x^3 + 17x^2 - 27x + 28 = 0$.
10. $3x^4 + 2x^3 + 3x^2 - 10x - 8 = 0$.
11. $12x^4 - 76x^3 + 73x^2 + 41x - 30 = 0$.
12. $x^4 - 5x^3 - 6x^2 + 10x + 56 = 0$.
13. $36x^4 - 4x^3 - 35x^2 + 3x + 6 = 0$.
14. $18x^4 + 15x^3 - 166x^2 + 47x + 140 = 0$.
15. $4x^4 + 20x^3 + 3x^2 - 34x - 7 = 0$.
16. $x^5 + 2x^4 + 8x^3 + 20x^2 + 5x = 0$.
17. $x^5 + x^2 + x + 1 = 0$.
18. $x^5 - x^4 - 5x^3 + 2x^2 + 15 = 0$.

Extract the following roots, correct to three decimal places, by using Horner's method or the method of section 115:

19. $\sqrt[3]{50}$.

Suggestion. Find the real root of the equation $x^3 = 50$.

20. $\sqrt[3]{17}$.
21. $\sqrt[5]{8}$.
22. $\sqrt[5]{113}$.

Solve the following simultaneous equations:

23. $x^2 + y = 11,$
$x + y^2 = 7.$

24. $x^4 + y^4 = 337,$
$x + y = 1.$

25. The edges of a rectangular box are 4, 6, and 10 inches, respectively. What equal increase of each dimension will increase the volume 100 cubic inches?

26. A sphere of radius r and specific gravity s, when floating in water, will sink to a depth x, which is a positive root of the equation $x^3 - 3rx^2 + 4r^3s = 0$. Find the depth to which a ball of radius 3 inches will sink if it is made of (a) cork (specific gravity 0.24), (b) white pine (specific gravity 0.45).

27. The edge of one cube is 4 inches longer than that of another. The combined volume of the two cubes is 1000 cubic inches. Find the edge of each.

28. A box with open top is to be constructed from a rectangular sheet of cardboard 12 by 20 inches by cutting equal squares from the corners and bending up the sides and ends. If the volume of the box is to be 200 cubic inches, what is the side of the square that must be cut out?

29. The height of a right circular cylinder is 3 inches greater than the diameter of its base. Its volume is 500 cubic inches. Find its dimensions.

30. A rectangular box is 5 feet long and 3 feet wide. A board 1 foot in width is fitted into the bottom of the box in the position shown in Fig. 34. Find the length of the board.

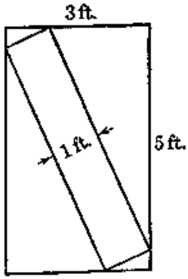


FIG. 34

The following cubic equations arose in designing reinforced concrete beams under bending and axial loads. The quantity x is the distance from the upper surface of the beam to the so-called neutral plane. Find the value of x correct to three decimal places.

31. $x^3 - 0.1629x^2 + 0.1960x - 0.1276 = 0.$

32. $x^3 + 0.0909x^2 + 0.1120x - 0.0904 = 0.$

33. $x^3 - 0.2562x^2 + 0.1140x - 0.1020 = 0.$

34. $x^3 - 0.3261x^2 + 0.1084x - 0.1024 = 0.$

35. $x^3 - 0.7743x^2 + 0.1364x - 0.1293 = 0.$
36. $x^3 - 0.3267x^2 + 0.1042x - 0.0989 = 0.$
37. $x^3 - 0.3972x^2 + 0.1019x - 0.0991 = 0.$
38. $x^3 - 0.8307x^2 + 0.1258x - 0.1240 = 0.$
39. $x^3 - 0.3048x^2 + 0.1018x - 0.0965 = 0.$
40. $x^3 - 0.3723x^2 + 0.0961x - 0.0936 = 0.$

119. The general cubic.

The general cubic equation is

$$a_0x^3 + a_1x^2 + a_2x + a_3 = 0, \qquad a_0 \neq 0, \tag{1}$$

or

$$x^3 + bx^2 + cx + d = 0, \tag{2}$$

where

$$b = \frac{a_1}{a_0}, \qquad c = \frac{a_2}{a_0}, \qquad d = \frac{a_3}{a_0}. \tag{3}$$

Set

$$x = y - \frac{b}{3}, \tag{4}$$

and (2) becomes

$$y^3 + Cy + D = 0, \tag{5}$$

in which

$$C = c - \frac{b^2}{3}, \qquad D = d - \frac{bc}{3} + \frac{2b^3}{27}. \tag{6}$$

Equation (5) is called the **reduced cubic.**

To solve the reduced cubic we introduce two unknowns, u and v, whose **sum** is to be a root of the reduced cubic; that is, we set

$$y = u + v, \tag{7}$$

and substitute in (5). The resulting equation may be placed in the form

$$u^3 + v^3 + (3uv + C)(u + v) + D = 0. \tag{8}$$

Since we have substituted two unknowns, u and v, for the single unknown y, we can impose a condition on them. If we impose the condition

$$3uv + C = 0, \tag{9}$$

equation (8) reduces to the simpler form

$$u^3 + v^3 + D = 0. \tag{10}$$

Solving (9) for v and substituting in (10), we obtain

$$u^3 - \frac{C^3}{27u^3} + D = 0, \tag{11}$$

or

$$u^6 + Du^3 - \frac{C^3}{27} = 0. \tag{12}$$

This is a quadratic in u^3, and we find

$$u^3 = -\frac{D}{2} \pm \frac{1}{2}\sqrt{D^2 + 4C^3/27}. \tag{13}$$

Taking

$$u^3 = -\frac{D}{2} + \frac{1}{2}\sqrt{D^2 + 4C^3/27}, \tag{14}$$

we find from (10) that

$$v^3 = -\frac{D}{2} - \frac{1}{2}\sqrt{D^2 + 4C^3/27}. \tag{15}$$

Note. If we take $u^3 = -\frac{D}{2} - \frac{1}{2}\sqrt{D^2 + 4C^3/27}$, we find that $v^3 = -\frac{D}{2} + \frac{1}{2}\sqrt{D^2 + 4C^3/27}$; i.e., u and v have simply been interchanged. This is to be expected, since they enter symmetrically into the various equations such as (7), (8), and (9).

There are three cube roots of (14) and three cube roots of (15). They can be found by the methods of section 101. Let U and V be cube roots of (14) and (15) respectively which satisfy condition (9), namely, $3UV = -C$. The other cube roots of (14) are ωU and $\omega^2 U$, and the other cube roots of (15) are ωV and $\omega^2 V$, where ω and ω^2 are the imaginary cube roots of 1 (see exercise XII. E, 23), namely,

$$\omega = -\frac{1}{2} + \frac{i}{2}\sqrt{3}, \qquad \omega^2 = -\frac{1}{2} - \frac{i}{2}\sqrt{3}.$$

These roots may be paired to satisfy (9) as follows:

$$3 \cdot \omega U \cdot \omega^2 V = -C, \qquad 3 \cdot \omega^2 U \cdot \omega V = -C.$$

If these pairs of values are substituted in (7), we obtain, for the roots of the reduced cubic,

$$y_1 = U + V, \quad y_2 = \omega U + \omega^2 V, \quad y_3 = \omega^2 U + \omega V. \tag{16}$$

Making use of (4) and (3), we find for the roots of (1),

$$\begin{aligned} x_1 &= U + V - \frac{a_1}{3a_0}, \\ x_2 &= \omega U + \omega^2 V - \frac{a_1}{3a_0}, \\ x_3 &= \omega^2 U + \omega V - \frac{a_1}{3a_0}. \end{aligned} \tag{17}$$

These expressions for the roots of a cubic are called **Cardan's formulas.**

Example 1.

Solve the equation $x^3 + 3x^2 + 15x + 1 = 0.$ (18)

SOLUTION. Set $x = y - 1$; (19)

$$y^3 + 12y - 12 = 0. \tag{20}$$

Set $y = u + v$ in (20):

$$u^3 + v^3 + (3uv + 12)(u + v) - 12 = 0. \tag{21}$$

Impose the condition

$$3uv + 12 = 0, \qquad \text{or} \qquad uv = -4, \tag{22}$$

reducing (21) to

$$u^3 + v^3 - 12 = 0. \tag{23}$$

Eliminate v between (22) and (23):

$$u^6 - 12u^3 - 64 = 0. \tag{24}$$

$$u^3 = 16. \tag{25}$$

From (22),

$$v^3 = -4. \tag{26}$$

Take

$$U = \sqrt[3]{16} = 2\sqrt[3]{2}, \qquad V = -\sqrt[3]{4}. \tag{27}$$

These values satisfy (22).

$$y_1 = 2\sqrt[3]{2} - \sqrt[3]{4}, \; y_2 = \omega 2\sqrt[3]{2} - \omega^2\sqrt[3]{4}, \; y_3 = \omega^2 2\sqrt[3]{2} - \omega\sqrt[3]{4};$$

$$x_1 = 2\sqrt[3]{2} - \sqrt[3]{4} - 1, \; x_2 = \omega 2\sqrt[3]{2} - \omega^2\sqrt[3]{4} - 1,$$

$$x_3 = \omega^2 2\sqrt[3]{2} - \omega\sqrt[3]{4} - 1.$$

Example 2.

Solve the equation $x^3 - 12x - 8 = 0.$ (28)

SOLUTION. Set $x = u + v$ in (28):

$$u^3 + v^3 + (3uv - 12)(u + v) - 8 = 0. \tag{29}$$

Impose the condition

$$3uv - 12 = 0, \qquad \text{or} \qquad uv = 4, \tag{30}$$

reducing (29) to

$$u^3 + v^3 - 8 = 0. \tag{31}$$

Eliminate v between (30) and (31):

$$u^6 - 8u^3 + 64 = 0. \tag{32}$$

$$u^3 = 4 + 4\sqrt{3} \cdot i = 8 \text{ cis } 60°. \tag{33}$$

From (30), $$v^3 = 4 - 4\sqrt{3} \cdot i = 8 \text{ cis } (-60°). \tag{34}$$

As cube roots of (33) and (34) respectively take

$$U = 2 \text{ cis } 20°, \qquad V = 2 \text{ cis } (-20°). \tag{35}$$

These values satisfy (30).

$$\begin{aligned} x_1 &= U + V = 4 \cos 20° = 3.759, \\ x_2 &= \omega U + \omega^2 V = 2 \text{ cis } 140° + 2 \text{ cis } 220° \\ &= -4 \cos 40° = -3.064, \\ x_3 &= \omega^2 U + \omega V = 2 \text{ cis } 260° + 2 \text{ cis } 100° \\ &= -4 \cos 80° = -0.695. \end{aligned}$$

120. The general quartic.

The general quartic, or fourth-degree equation is

$$a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 = 0, \tag{1}$$

or $$x^4 + bx^3 + cx^2 + dx + e = 0. \tag{2}$$

The first step in the solution of (2) is to write it in the form

$$x^4 + bx^3 = -cx^2 - dx - e. \tag{3}$$

Complete the square of the left side by adding $b^2x^2/4$ to both sides:

$$x^4 + bx^3 + \frac{b^2x^2}{4} = \left(\frac{b^2}{4} - c\right) x^2 - dx - e. \tag{4}$$

We now introduce a number y, whose value is to be determined later, and add $\left(x^2 + \frac{bx}{2}\right)y + \frac{y^2}{4}$ to both sides of (4), getting

$$\left(x^2 + \frac{bx}{2}\right)^2 + \left(x^2 + \frac{bx}{2}\right)y + \frac{y^2}{4}$$
$$= \left(\frac{b^2}{4} - c + y\right)x^2 + \left(\frac{by}{2} - d\right)x + \frac{y^2}{4} - e. \quad (5)$$

The right member of (5) is quadratic in x and will be a perfect square if its discriminant is zero, that is, if

$$\left(\frac{by}{2} - d\right)^2 - 4\left(\frac{b^2}{4} - c + y\right)\left(\frac{y^2}{4} - e\right) = 0. \quad (6)$$

Equation (6) simplifies into*

$$y^3 - cy^2 + (bd - 4e)y - b^2e + 4ce - d^2 = 0. \quad (7)$$

We now determine y so that (7) will be satisfied; that is, we find a root of the cubic (7). Let this root be $y = r$. Substitute it in (5), whose right side will then be the square of some linear function, $px + q$. Equation (5) will then have the form

$$\left(x^2 + \frac{bx}{2} + \frac{r}{2}\right)^2 = (px + q)^2, \quad (8)$$

from which we get

$$x^2 + \frac{bx}{2} + \frac{r}{2} = \pm(px + q). \quad (9)$$

The two quadratic equations in (9) can be solved by the

* Equation (7) is called the **resolvent cubic** for (1), or (2)

quadratic formula or otherwise, yielding altogether four roots.

It can be shown that no matter which one of the three roots of (7) is used, the final results will be the same.

The foregoing method is due to **Ferrari.**

Example.

Solve the equation

$$x^4 - 4x^3 + 4x^2 - 12x + 3 = 0. \tag{10}$$

Solution.

$$x^4 - 4x^3 = -4x^2 + 12x - 3.$$

Add $4x^2$ to both sides:

$$x^4 - 4x^3 + 4x^2 = 12x - 3.$$

Add $(x^2 - 2x)y + \frac{y^2}{4}$ to both sides:

$$(x^2 - 2x)^2 + (x^2 - 2x)y + \frac{y^2}{4} = (x^2 - 2x)y + \frac{y^2}{4} + 12x - 3,$$

$$\left(x^2 - 2x + \frac{y}{2}\right)^2 = yx^2 - 2(y - 6)x + \frac{y^2}{4} - 3. \tag{11}$$

Equate to zero the discriminant of the right side:

$$4(y - 6)^2 - 4y\left(\frac{y^2}{4} - 3\right) = 0,$$

$$y^3 - 4y^2 + 36y - 144 = 0.$$

The roots of this last equation are $4, \pm 6i$.

Set $y = 4$ in (11):

$$(x^2 - 2x + 2)^2 = 4x^2 + 4x + 1.$$

$x^2 - 2x + 2 = 2x + 1,$	$x^2 - 2x + 2 = -(2x + 1),$
$x^2 - 4x + 1 = 0,$	$x^2 = -3,$
$x = 2 \pm \sqrt{3}.$	$x = \pm i\sqrt{3}.$

EXERCISES XIII. J

Solve:

1. $x^3 - 9x - 12 = 0.$
2. $x^3 - 12x + 20 = 0.$
3. $x^3 - 6x^2 + 24 = 0.$
4. $x^3 + 9x^2 + 36x + 28 = 0.$
5. $x^3 - 3x^2 - 12x - 16 = 0.$
6. $x^3 - 3x - \sqrt{2} = 0.$
7. $x^3 - 9x^2 + 21x - 5 = 0.$
8. $x^3 - 3x^2 - 144x - 2304 = 0.$
9. $x^4 + 8x^3 + 20x^2 + 16x - 21 = 0.$
10. $x^4 - 6x^3 + 7x^3 + 6x - 2 = 0.$
11. $x^4 - 4x^3 + 5x^2 - 16x + 4 = 0.$
12. $x^4 + 2x^3 - 20x^2 - 6x + 3 = 0.$
13. $x^4 + 6x^3 - 26x + 15 = 0.$
14. $x^4 + 4x^3 + 3x^2 - 14x - 9 = 0.$
15. A wall 4 feet high is parallel to, and at a distance of 4 feet from, the face of a building. A ladder 28 feet long is leaned against the face of the building and is placed in such a position that it just touches the top of the wall. How high up the building does it reach? (It is assumed that the ground is level and that the ladder is in a vertical plane which is perpendicular to the face of the building.)

121. Algebraic solution of equations.

The solutions of an integral rational equation,

$$a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0, \qquad (1)$$

are functions of the coefficients. If these functions involve no operations other than a finite number of additions, subtractions, multiplications, divisions, and extractions of roots, the solution is **algebraic.**

We have already obtained algebraic solutions of the general quadratic, cubic, and quartic ($n = 2, 3, 4$, respectively, in (1)), and it might be surmised that the general equation of any degree could be similarly solved. However, it can be proved that the general equation of degree higher than four has no algebraic solution.

122. Coefficients in terms of roots.

If $r_1, r_2, \cdots, r_n$ are the roots of an equation, then the equation may be written

$$(x - r_1)(x - r_2) \cdots (x - r_n) = 0. \tag{1}$$

Multiplying out, we get

$$\begin{aligned} x^n &- (r_1 + r_2 + \cdots + r_n)x^{n-1} \\ &+ (r_1r_2 + r_1r_3 + \cdots + r_1r_n + r_2r_3 + \cdots + r_{n-1}r_n)x^{n-2} \\ &- (r_1r_2r_3 + \cdots + r_{n-2}r_{n-1}r_n)x^{n-3} + \cdots \\ &+ (-1)^n r_1r_2 \cdots r_n = 0. \end{aligned} \tag{2}$$

Comparing (2) with the form

$$x^n + b_1x^{n-1} + b_2x^{n-2} + b_3x^{n-3} + \cdots + b_{n-1}x + b_n = 0, \tag{3}$$

we see that

$b_1 = -$*sum of roots,*
$b_2 =$ *sum of products of roots taken two at a time,*
$b_3 = -$*sum of products of roots taken three at a time,*
. . . .
$b_n = (-1)^n \times$ *product of roots.*

If the equation is given in the form

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n = 0, \tag{4}$$

it is merely necessary to divide through by a_0 to reduce it to the form (3).

EXERCISES XIII. K

Find the sum of the roots and the product of the roots of the following equations:

1. $x^3 + 6x^2 + 4x + 5 = 0.$
2. $3x^3 - 2x^2 - 4x + 9 = 0.$

3. $6x^4 - 9x^3 - 4x^2 - 7x + 12 = 0$.
4. $3x^4 + 5x^3 - 17x - 30 = 0$.
5. $x^5 + 4x^4 + 3x^3 + 2x^2 + 8x + 9 = 0$.
6. $4x^6 - 2x^5 + 7x^3 - x - 1 = 0$.
7. $\frac{1}{4}x^4 - \frac{1}{3}x^3 + \frac{1}{2}x^2 - \frac{1}{8}x + \frac{1}{6} = 0$.
8. Solve the equation $x^3 - 5x^2 + kx + 36 = 0$, given that the sum of two of its roots is 3. What must be the value of k?
9. Solve the equation $2x^3 - 25x^2 + kx + 132 = 0$, given that the product of two of its roots is 44. Find the value of k.
10. Solve the equation $3x^3 + 4x^2 + kx - 9 = 0$, given that one root is the negative of another. Find k.
11. Solve the equation $9x^3 - 6x^2 + kx - 8 = 0$, given that it has a double root. Find k.
12. Solve the equation $5x^3 + kx^2 + 16x - 10 = 0$, given that two of its roots are the negative reciprocals of each other. Find k.
13. Solve the equation $12x^4 + 4x^3 + cx^2 + dx + 75 = 0$, given that it has two double roots. Find c and d.
14. Solve the equation $x^4 - 12x^3 + cx^2 + dx + 121 = 0$, given that it has an imaginary double root. Find c and d.
15. Solve the equation $6x^4 - 21x^3 + 20x^2 + dx + e = 0$, given that it has a triple root. Find d and e.
16. Solve the equation $x^3 + kx^2 + 5x + 12 = 0$, given that the sum of two of its roots is 7. Find k.
17. Show that if the equation $x^3 + cx + d = 0$ has a double root, then $\left(\frac{c}{3}\right)^3 + \left(\frac{d}{2}\right)^2 = 0$, and conversely.
18. Show that if the equation $x^3 + bx^2 + cx + d = 0$ has a triple root, then $d = \left(\frac{c}{b}\right)^3$, and conversely.

CHAPTER XIV

Logarithms

123. Logarithm.

The **logarithm** of a number to a given base is the exponent of the power to which the base must be raised to give the number. (It is assumed that the base is positive and different from 1, and that the number is positive.) Thus, since $2^3 = 8$, the exponent 3 is the logarithm of 8 to the base 2. This may be written in the form $3 = \log_2 8$. More generally, if $N = b^x$, we write $x = \log_b N$.

The base in most common use is 10. Since, for example, $10^2 = 100$, we have $\log_{10} 100 = 2$. As the next few sections deal with logarithms to the base 10, we shall, for the present, omit the subscript indicating that 10 is the base, and write simply $\log 100 = 2$.

Similarly, $10^{-2} = 0.01$, and we have $\log 0.01 = -2$.

Thus, we can construct the following table, which shows both the exponential and the logarithmic form of writing the same relation.

$$\begin{array}{rlrl} 10^3 = & 1000, & \log 1000 = & 3; \\ 10^2 = & 100, & \log 100 = & 2; \\ 10^1 = & 10, & \log 10 = & 1; \\ 10^0 = & 1, & \log 1 = & 0; \\ 10^{-1} = & 0.1, & \log 0.1 = & -1; \\ 10^{-2} = & 0.01, & \log 0.01 = & -2; \\ 10^{-3} = & 0.001, & \log 0.001 = & -3. \end{array}$$

Further, since $10^{1/2} = \sqrt{10}$, we have $\log \sqrt{10} = \frac{1}{2} = 0.5$. Likewise, $\log \sqrt[3]{10} = \frac{1}{3}$.

However, there is no rational number x for which $10^x = 3$, for example, and we have not yet defined a^x for irrational values of x. We assume that when x is an irrational number, and a is positive and greater than 1, a^x is a number which is greater than a^y and less than a^z, where y and z are rational numbers which are respectively less than and greater than x. That is,*

$$a^y < a^x < a^z, \qquad \text{where} \qquad y < x < z.$$

We make the further assumption that as y and z are taken closer and closer to x, a^y and a^z assume values which are closer and closer together. In fact we define a^x as the common limit of a^y and a^z, as y and z approach x through sequences of rational values. With the foregoing assumptions, we may write the approximate relation

$$10^{0.4771} = 3, \qquad \text{or} \qquad \log 3 = 0.4771.$$

This means that the four-digit number which is closest to log 3 is 0.4771, just as the four-digit number which is closest to $\sqrt{2}$ is 1.414. The method of obtaining this approximation to log 3 involves more advanced mathematics than the familiar method of extracting square roots used in finding the approximation to $\sqrt{2}$, and will not be explained here.

124. Mantissa.

Expressing the relation $10^{1/3} = \sqrt[3]{10} = 2.154$ in logarithmic form, we have (approximately of course)

$$\log 2.154 = \frac{1}{3} = 0.3333.$$

If we take the relation

$$10^{0.3333} = 2.154 \tag{1}$$

* If $0 < a < 1$, then $a^y > a^x > a^z$, where $y < x < z$.

and multiply both sides by 10, we get

$$10^{1.3333} = 21.54,$$

which in logarithmic notation is

$$\log 21.54 = 1.3333.$$

By dividing both sides of (1) by 10 we get

$$10^{0.3333-1} = 0.2154,$$

or *

$$\log 0.2154 = 0.3333 - 1.$$

These two examples illustrate the fundamental principle: *For numbers having the same sequence of digits*, such as 2.154, 215.4, 0.002,154, *the decimal part of the logarithm (called the* ***mantissa****) is the same.*† The method of finding the mantissa from tables will be given in a later section.

125. Characteristic.

The whole-number part of the logarithm is called the **characteristic.** Thus, since log 21.54 = 1.3333, the characteristic of the logarithm of 21.54 is 1.

We know that

$$10^0 = 1, \qquad 10^1 = 10,$$

or

$$\log 1 = 0, \qquad \log 10 = 1.$$

Thus, logarithms of numbers between 1 and 10, such as 2.154, have the characteristic 0. For such numbers we shall say that the decimal point is in **standard position,** namely, immediately after the first non-zero digit.

Each time we multiply a number by 10 we move the decimal point one place to the right, and each time we

* This could also be written log 0.2154 = −0.6667, but it is usually more convenient to keep the decimal part of a logarithm positive. (But see section 129, example 4.)

† Provided that the base is 10.

divide a number by 10 we move the point one place to the left. But each time we multiply by 10 we increase the logarithm of the number by 1, and each time we divide a number by 10 we decrease its logarithm by 1, as was seen in the illustration above. Thus, we may state the following rule for finding characteristics:

If a number has its decimal point in standard position (i.e., after the first non-zero digit), *the characteristic of the logarithm of the number is zero; if the decimal point is not in standard position, the characteristic is equal to the number of places the point has been moved, and is positive if the point has been moved to the right, negative if it has been moved to the left.*

For example, in the number 78,460, the decimal point has been moved from standard position (after the 7) 4 places to the right (after the 0), and the characteristic of the logarithm of 78,460 is therefore 4.

In the number 0.02154, the point has been moved from standard position (after the 2) 2 places to the left. The characteristic of the logarithm of the number is -2. In fact, since we saw above that $\log 2.154 = 0.3333$, we may write

$$\log 0.02154 = 0.3333 - 2.$$

This may also be written

$$\log 0.02154 = 8.3333 - 10 \qquad (\text{since } 8 - 10 = -2),$$

a form frequently used. Another form is

$$\log 0.02154 = \bar{2}.3333$$

(*not* -2.3333, as this would mean that the entire logarithm is negative, whereas only the characteristic is negative, this being indicated by the minus sign above the characteristic). This form is not recommended.

The rule given for determining the characteristic also tells us how to point off a number corresponding to a given logarithm. (The number corresponding to a logarithm is called the **antilogarithm.**)

Thus, if we have given

$$\log N = 2.3333,$$

we know from the illustration used above that the number N is composed of the sequence of digits 2154. Since the characteristic is 2, the decimal point has been moved 2 places to the right from standard position. Therefore

$$N = 215.4.$$

EXERCISES XIV. A

Determine the characteristic of the logarithm of.

1. 328.4. **2.** 18.96. **3.** 13000.
4. 3.982. **5.** 0.5623. **6.** 0.09843.
7. 69470. **8.** 747000. **9.** 0.000,312,4.
10. 27. **11.** 27.83. **12.** 2.7.
13. 0.0015. **14.** 3. **15.** 3.000.
16. 4004. **17.** 4.004. **18.** 0.000003.
19. 0.0008. **20.** 1.0008. **21.** 80.00.
22. 0.5. **23.** 0.5000. **24.** 0.000,000,5.

Given $\log 7.28 = 0.8621$; find the value of x for which

25. $\log x = 2.8621$. **26.** $\log x = 1.8621$.
27. $\log x = 0.8621 - 2$. **28.** $\log x = 9.8621 - 10$.
29. $\log x = 4.8621$. **30.** $\log x = 4.8621 - 10$.
31. $\log x = 0.8621 - 4$. **32.** $\log x = 3.8621$.

Given $\log 423 = 2.6263$; find the value of x for which

33. $\log x = 0.6263$. **34.** $\log x = 0.6263 - 1$.
35. $\log x = 8.6263 - 10$. **36.** $\log x = 5.6263$.
37. $\log x = 5.6263 - 10$. **38.** $\log x = 0.6263 - 5$.
39. $\log x = 10.6263 - 10$. **40.** $\log x = 7.6263 - 10$.

126. Finding the mantissa.

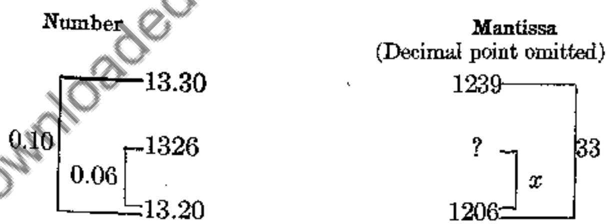
Tables of logarithms give mantissas (decimal point omitted). They are called four-place tables, five-place tables, and so on, according to the number of digits in the mantissas. Table II at the end of the book is a four-place table. In it the first two digits of a number are found at the left of the page, the third digit at the top, the corresponding mantissa being in the same row as the first two digits of the number, and in the same column as the third digit of the number. Thus, to find the mantissa of the logarithm of 132, we follow across the row which has 13 at the left until we come to the column headed by 2. In this position we find the mantissa 1206. Since the characteristic of the logarithm of 132 is 2, the complete logarithm is 2.1206.

To find, from Table II, the logarithm of a four-digit number, we use a process called **interpolation,** illustrated below.

Example.

Find log 13.26.

Solution. Find the mantissas for the numbers next above and below 13.26:



Assuming that the change in the mantissa is proportional to the change in the number,* we have

$$\frac{x}{33} = \frac{0.06}{0.10} = 0.6,$$
$$x = 0.6 \times 33 = 19.8.$$

* This is only approximately true.

When interpolating in a four-place table we retain only four digits in the mantissa. Thus we add 20, not 19.8.

Mantissa = 1206 + 20 = 1226.

$$\log 13.26 = 1.1226.$$

Once the principle of proportionality, or proportional parts, is understood, the work can be arranged more compactly as follows, or may be performed mentally.

$$\begin{array}{rr} 13.30 \sim & 1239 \\ 13.20 \sim & \underline{1206} \\ \text{difference} = & 33 \\ & \underline{\times\, 0.6} \\ & 19.8 \\ & \underline{1206} \\ \log 13.26 = & 1.1226 \end{array}$$

(The symbol $\sim$ may here be read " corresponds to.")

To find, from Table II, the logarithm of a number composed of more than four digits, we first round off the number to four digits. To **round off** a number to k digits (or k figures, or k places*) means to find the closest approximation to the number that can be written with k digits. For example, we know that, to five digits, $\pi = 3.1416$. Rounded off to four digits, the approximation is 3.142, this being closer to the true value of π than 3.141, or any other four-digit number. The number 357,238, rounded off to four digits, becomes 357,200. (See section 129.)

When the rounding-off process can lead to two numbers, each equally close to the given number, we shall adopt the arbitrary rule of choosing *the one which ends in an even digit.* For example, 13.425 becomes 13.42, while 286.35 becomes 286.4.

* Not decimal places.

EXERCISES XIV. B

1. Find log 29.17.

SOLUTION.

$$\begin{array}{rr} 29.20 \sim & 4654 \\ 29.10 \sim & \underline{4639} \\ \text{difference} = & 15 \\ & \underline{\times\, 0.7} \\ & 10.5 \end{array}$$

Here we may add either 10 or 11 to 4639. However, in a situation of this kind, we follow the rule given above and add 11, so as to make the resulting number even. Thus,

$$\begin{array}{rr} & 4639 \\ & \underline{11} \\ \log 29.17 = & 1.4650 \end{array}$$

Find the logarithm of each of the following numbers:

2. 264.	**3.** 3.	**4.** 3.25.
5. 400.	**6.** 20.	**7.** 0.632.
8. 0.00413.	**9.** 1.935.	**10.** 831.2.
11. 37.85.	**12.** 6193.	**13.** 79870.
14. 0.5506.	**15.** 0.07393.	**16.** 1004.
17. 0.9997.	**18.** 0.007093.	**19.** 8,881,000.
20. 549900.	**21.** 0.01001.	**22.** 0.9009.

127. Finding the antilogarithm.

The process of finding the number corresponding to a given logarithm is illustrated by the following examples:

Example 1.

Find the number whose logarithm is 7.8414 − 10.

SOLUTION. The mantissa 8414 is found in Table II. At the left we find 69, at the top we find 4. Thus the number is 0.00694.

Example 2.

Given $\log N = 1.8418$; find N.

Solution. Here we use inverse interpolation, finding the mantissas next above and next below 8418.

$$\frac{x}{1} = \frac{4}{6} = 0.7,$$
$$N = 69.47.$$

Note. *The process of interpolation, including inverse interpolation, is applicable to any kind of table*, e.g., a table of square roots, provided of course that the values given in the table are sufficiently close together.

EXERCISES XIV. C

Find the number corresponding to each of the following logarithms:

1. 0.5465.	**2.** 2.8494.	**3.** 9.3139 − 10.
4. 8.9390 − 10.	**5.** 0.7782.	**6.** 1.8994.
7. 2.2236.	**8.** 7.6700 − 10.	**9.** 3.8072.
10. 4.4143.	**11.** 1.5000.	**12.** 9.9998 − 10.
13. 1.0200.	**14.** 0.1020.	**15.** 0.0102.
16. 2.0019.	**17.** 8.5369 − 10.	**18.** 5.8451.
19. 3.5350.	**20.** 0.3535.	**21.** 0.0353.
22. 8.8750.	**23.** 7.7400 − 10.	**24.** 4.0255.

128. Laws of logarithms.

Since logarithms are exponents, they obey the laws of exponents, it being assumed that these laws hold for irrational as well as rational exponents. Thus:

I. The logarithm of a product is equal to the sum of the logarithms of its factors.

Let $\log_b = x, \qquad \log_b N = y.$

Then, $M = b^x, \qquad N = b^y.$

$$MN = b^x b^y = b^{x+y}.$$

$$\log_b MN = x + y,$$

or $\mathbf{\log_b MN = \log_b M + \log_b N.}$

The proof can easily be extended to cover the case of any finite number of factors.

Example.

$$\log (214 \cdot 386) = \log 214 + \log 386.$$

II. The logarithm of a quotient is equal to the logarithm of the dividend minus the logarithm of the divisor.

Using the same notation as above, we have

$$\frac{M}{N} = \frac{b^x}{b^y} = b^{x-y}.$$

$$\log_b \frac{M}{N} = x - y.$$

$$\mathbf{\log_b \frac{M}{N} = \log_b M - \log_b N.}$$

Example.

$$\log \frac{214}{386} = \log 214 - \log 386.$$

III. The logarithm of a power of a number is equal to the exponent of the power times the logarithm of the number.

If $\log_b N = x$, then $N = b^x$, and

$$N^m = (b^x)^m = b^{mx}.$$

$$\log_b N^m = mx.$$

$$\mathbf{\log_b N^m = m \log_b N.}$$

Example.

$$\log (2.14)^3 = 3 \log (2.14).$$

IV. The logarithm of a real positive root of a number is equal to the logarithm of the number divided by the index of the root.

This is really the same as III, since $\sqrt[m]{N} = N^{1/m}$. Thus,

$$\log_b \sqrt[m]{N} = \frac{1}{m} \log_b N.$$

Example.

$$\log \sqrt[5]{3} = \frac{1}{5} \log 3.$$

Several of these laws may of course be involved together.

Example.

$$\log \frac{17.6(2.31)^3}{986\sqrt{18.7}} = \log 17.6 + 3 \log 2.31 - \log 986 - \tfrac{1}{2} \log 18.7,$$

$$\log \frac{4}{3} \pi r^3 = \log 4 + \log \pi + 3 \log r - \log 3,$$

$$2 \log A - 3 \log B + \frac{1}{3} \log C = \log \frac{A^2 \sqrt[3]{C}}{B^3}.$$

129. Computation with logarithms.

The advantages of logarithms in computation are that multiplication and division can be replaced by the simpler operations of addition and subtraction respectively, and that raising to powers and extracting roots can be replaced by multiplication and division.

It must be realized that logarithms, with a few exceptions, are approximations, and that results obtained by using them are only approximate. It therefore seems advisable, before illustrating their use in computation, to give a brief discussion of the accuracy to be expected when operating with approximate numbers.

The **significant digits** (or figures) in the decimal form of a number are the digits reading from left to right beginning with the first non-zero digit and ending with the last digit written. (But see note below.) Observe that the number of significant digits does not depend on the position of the decimal point.

The approximate number 7.2 has two significant digits. The number 7.20 has three significant digits. The true value of the quantity which it represents is between 7.195 and 7.205, whereas the true value represented by the approximate number 7.2 is between 7.15 and 7.25. That is, 7.20 is a closer approximation than 7.2. In general, the greater the number of significant digits in an approximate number, the more accurate the number.

NOTE. Final 0's in a whole number may or may not be significant. For example, if it is stated that the population of a town is 15,000, it is impossible to tell which of the 0's, if any, are significant. If the population is given to the nearest thousand, none of them is; if it is given to the nearest hundred, the first 0 is significant, the other two are not. (To indicate in these two cases which digits are significant we might write $15 \cdot 10^3$ and $150 \cdot 10^2$ respectively.)

Results obtained by using approximate numbers are usually no more accurate than the least accurate number entering into the computation. By way of illustration, suppose that the sides of a rectangle have been measured as 13.7 feet and 10.2 feet respectively. Multiplying the two dimensions together, we should find for the area 139.74 square feet. However, not all of the figures in this result are accurate. For if we recall the meanings of the approximate numbers 13.7 and 10.2, we see that the actual number of square feet in the area may be between $13.65 \times 10.15 = 138.5475$ and $13.75 \times 10.25 = 140.9375$. Therefore, to say that the area is 139.74 square feet is to claim false accuracy. The number should be rounded off to 140 (that is, to three significant digits, the number of such digits in each of the numbers multiplied together). Even then, there may be an error in the last digit.

A four-place table of logarithms is adapted to use with numbers having four significant digits. In the following examples and exercises in computation we shall assume that any number with fewer than four places is exact, and shall round off to four digits any numbers of greater accuracy. (See section 126.) The results of computations will be assumed to contain just four significant digits, and should be correspondingly rounded off. In other words, results should be the best obtainable by using interpolation in Table II.

Example 1.

Find the value of $x = 36.2 \times 867$.

SOLUTION.

$$\begin{aligned} \log 36.2 &= 1.5587 \\ \log 867 &= \underline{2.9380} \\ \log x &= 4.4967 \\ x &= 31390. \end{aligned}$$

Example 2.

Find the value of $x = 36.2 \div 867$.

SOLUTION.

$$\begin{aligned} \log 36.2 &= 1.5587 \\ \log 867 &= \underline{2.9380} \end{aligned}$$

Here we are subtracting the larger quantity from the smaller. In order to keep the mantissa positive, we add 2 to, and subtract 2 from, the logarithm of 36.2, getting

$$\begin{aligned} \log 36.2 &= 3.5587 - 2 \\ \log 867 &= \underline{2.9380 \qquad} \\ \log x &= 0.6207 - 2 \\ x &= 0.04175. \end{aligned}$$

Some prefer to add and subtract 10 instead of 2; thus:

$$\begin{aligned} \log 36.2 &= 11.5587 - 10 \\ \log 867 &= \underline{2.9380} \\ \log x &= 8.6207 - 10 \\ x &= 0.04175. \end{aligned}$$

Example 3.

Find the value of $x = (0.362)^4$.

SOLUTION.

$$\begin{aligned} \log 0.362 &= 0.5587 - 1 \\ &\quad\ \underline{\times 4} \\ \log x &= 2.2348 - 4 = 0.2348 - 2 \\ x &= 0.01717. \end{aligned}$$

Example 4.

Find the value of $x = (0.362)^{-4}$.

SOLUTION. $\log 0.362 = 0.5587 - 1 = -0.4413$

$$\begin{aligned} &\underline{\times -4} \\ \log x &= 1.7652 \\ x &= 58.24. \end{aligned}$$

Example 5.

Find the cube root of 36.2

SOLUTION.

$$\begin{aligned} \log 36.2 &= \underline{1.5587}(\div 3 \\ \log \sqrt[3]{36.2} &= 0.5196 \\ \sqrt[3]{36.2} &= 3.308. \end{aligned}$$

Example 6.

Find the cube root of 0.362.

SOLUTION. $\log 0.362 = 0.5587 - 1.$

In order to make the negative part of the characteristic exactly divisible by 3, add 2 and subtract 2:

$$\begin{aligned} \log 0.362 &= \underline{2.5587 - 3}(\div 3 \\ \log \sqrt[3]{0.362} &= 0.8529 - 1 \\ \sqrt[3]{0.362} &= 0.7127. \end{aligned}$$

Example 7.

Find the value of $x = \dfrac{0.156(3.62)^3}{86.7\sqrt[5]{918}}$.

SOLUTION.

$$\begin{aligned} \log 3.62 &= 0.5587 \\ &\quad\ \underline{\times 3} \\ &\quad\ 1.6761 \\ \log 0.156 &= \underline{0.1931 - 1} \\ \log \text{numerator} &= 2.8692 - 2 \\ \log \text{denominator} &= \underline{2.5306} \\ \log x &= 0.3386 - 2 \\ x &= 0.02181. \end{aligned}$$

$$\begin{aligned} \log 918 &= \underline{2.9628}(\div 5 \\ &\quad\ 0.5926 \\ \log 86.7 &= \underline{1.9380} \\ \log \text{denominator} &= 2.5306 \end{aligned}$$

NOTE. In a problem such as this it is a good plan to make an outline before proceeding with the actual computation.

Example 8.

Find the value of $\dfrac{0.156(-3.62)^3}{-86.7\sqrt[5]{-918}}$.

SOLUTION. We treat the example as if all numbers involved were positive, and then prefix the proper sign to the result. Here we have symbolically

$$\frac{+\cdot(-)^3}{-\cdot\sqrt[5]{-}} = \frac{+\cdot -}{-\cdot -} = -.$$

The actual logarithmic work would be precisely like that in example 7; the final result however would be -0.02181.

130. Cologarithm.

The logarithm of the reciprocal of a number is called the **cologarithm** of the number and is abbreviated **colog.** That is,

$$\text{colog } N = \log \frac{1}{N} = \log 1 - \log N = -\log N.$$

Thus, *the cologarithm of a number is the negative of the logarithm of the number*. Consequently, in solving a problem in division by means of logarithms we may either subtract the logarithm of the divisor or add its cologarithm. There is no advantage, but rather a disadvantage, in using the cologarithm when only two numbers are involved in a division problem. There is, however, some advantage, particularly in the arrangement of the solution, when more than one number occurs in the denominator of a fractional expression.

The cologarithm is obtained by finding the logarithm and subtracting it from log 1, that is, from 10 − 10, which is of course 0. This can be done mentally after some practice.

Examples.

$\log 30.1 = 1.4786,$
$\text{colog } 30.1 = 8.5214 - 10.$

$\log 0.0375 = 8.5740 - 10,$
$\text{colog } 0.0375 = 1.4260.$

The following example illustrates the use of cologarithms:

Example.

Find the value of $x = \dfrac{0.584}{30.1 \times 0.0375}$.

SOLUTION.

$$\begin{aligned} \log 0.584 &= 9.7664 - 10 \\ \text{colog } 30.1 &= 8.5214 - 10 \\ \text{colog } 0.0375 &= 1.4260 \\ \hline \log x &= 19.7138 - 20 \\ x &= 0.5174. \end{aligned}$$

EXERCISES XIV. D

Find, by means of logarithms, the value of each of the following expressions:

1. 125.0×54.43.
2. 3.262×0.7809.
3. $68.41 \div 5.317$.
4. $23.72 \div 452.7$.
5. 0.4506×0.06503.
6. 1658×1319.
7. $0.2183 \div 0.9794$.
8. $0.6926 \div 0.03373$.
9. $4.781 \div 0.8693$.
10. $6.604 \div 3205$.
11. $0.3521 \div 0.04387$.
12. $0.08908 \div 0.5805$.
13. $5.689 \times 0.9987 \times 60.53$.
14. $0.2176 \times 0.4308 \times 1134$.
15. $\dfrac{85.91 \times 5.277}{496.2}$.
16. $\dfrac{74.23}{27.28 \times 0.3964}$.
17. $\dfrac{53.62 \times 9.248}{882.1 \times 0.9735}$.
18. $\dfrac{0.3732 \times 7668}{8464 \times 0.01916}$.
19. $(4.127)^2$.
20. $(0.5268)^2$.
21. $(0.06068)^3$.
22. $720.2(3.377)^2$.
23. $\dfrac{16.45}{(8.552)^2}$.
24. $\dfrac{(0.3773)^2}{4.586}$.
25. $\dfrac{(77.15)^3}{(207.3)^2}$.
26. $\dfrac{609.9(0.5428)^2}{(41.51)^3}$.
27. $\pi(0.5037)^2$.
28. $\frac{4}{3}\pi(5.867)^3$.
29. $\sqrt{42.01}$.
30. $\sqrt{0.1386}$.
31. $\sqrt{25.28 \times 0.3143}$.
32. $7.261\sqrt{0.01148}$.
33. $\sqrt[3]{1.786}$.
34. $\sqrt[3]{0.3847}$.
35. $\sqrt{\dfrac{4.533}{93.88}}$.
36. $\dfrac{\sqrt[3]{0.2765}}{\sqrt{6.513}}$.
37. $\dfrac{295.6\sqrt{13.97}}{5807}$.
38. $\dfrac{5636(0.7595)^2}{\sqrt{3153}}$.
39. $\dfrac{-23.22(-9.456)}{(-391.8)^2}$.
40. $\dfrac{825.5(-0.3623)^3}{1942}$.
41. $\dfrac{\sqrt{-0.7882}}{-8.838}$.
42. $\dfrac{\sqrt[3]{-713.5}}{-91.07}$.
43. $(1.005)^{-4}$.
44. $(0.9843)^{-2}$.
45. $(1.004)^{-1/3}$.
46. $(0.05196)^{-2/7}$.
47. $(\frac{2}{3})^{2/3}$.
48. $(\frac{3}{5})^{-3/5}$.

49. $\dfrac{4.536}{17.92}$.
50. $\dfrac{\log 4.536}{\log 17.92}$.

51. $\dfrac{\log 3}{\log 7}$.
52. $\dfrac{\log 673.9}{673.9}$.

53. $\dfrac{\log 0.6984}{\log 4.777}$.
54. $\dfrac{\log 0.02436}{\log 0.6856}$.

55. $\log 37.94 \times \log 2.743$.
56. $\log 1.807 \times \log 0.3071$.

57. $\sqrt{\log 10.76}$.
58. $\sqrt{\log 0.4521}$.

59. Find the area of a triangle whose base is 31.98 inches and whose altitude is 12.12 inches.

60. Find (a) the circumference and (b) the area of a circle whose radius is 0.5428 inch.

61. Find the volume of a sphere whose radius is 7.274 inches. ($V = \frac{4}{3}\pi r^3$.)

62. What is the weight of a steel ball bearing which is $\frac{3}{16}$ of an inch in diameter, if steel weighs 485 pounds per cubic foot?

63. Find the volume of a right circular cone whose height is 88.34 centimeters and the radius of whose base is 28.98 centimeters.

64. Neglecting air resistance, the velocity, in feet per second, acquired by an object in falling through a distance of s feet is given by the formula $v = \sqrt{2gs}$, in which g has the value 32.16 feet per second per second. Find the velocity acquired by an object in falling 1225 feet.

65. The period of a simple pendulum (the time in seconds required for it to swing across and back) is given by the formula

$$t = 2\pi\sqrt{\frac{l}{g}},$$

in which l is the length in feet and g has the value given in the preceding exercise. Find the period of a pendulum 3 feet 5 inches long.

66. The absolute pressure P and the volume V of a given weight of steam are connected by the approximate relation $PV^{1.06} = c$, in which c is a constant. Taking $c = 11480$, calculate (a) the value of P when $V = 38.42$, (b) the value of V when $P = 135.7$.

67. The absolute pressure P and the absolute temperature T of a given weight of gas are connected by the equation

$TP^{(1-k)/k} = c$, in which k and c are constants. Taking $k = 1.3$ and $c = 288.1$, calculate (a) the value of T corresponding to $P = 102.5$, (b) the value of P corresponding to $T = 830.6$.

131. Exponential equations.

An **exponential equation** is one involving an unknown or unknowns in an exponent, for example, $3^x = 5$, $2^{x+y} = 3^{2x}$. Logarithms are essential in the solution of such equations.

Example 1.

Solve the equation $3^x = 5$.

SOLUTION. Take logarithms of both sides:

$$x \log 3 = \log 5,$$
$$x = \frac{\log 5}{\log 3} = \frac{0.6990}{0.4771} = 1.465+.$$

Example 2.

Solve the equation $3^x \cdot 2^{3x-1} = 6^{2x+1}$.

SOLUTION. Take logarithms:

$$x \log 3 + (3x - 1) \log 2 = (2x + 1) \log 6,$$
$$x\,(\log 3 + 3 \log 2 - 2 \log 6) = \log 6 + \log 2,$$
$$x \log \frac{3 \cdot 2^3}{6^2} = \log 12,$$
$$x = \frac{\log 12}{\log \frac{2}{3}} = \frac{\log 12}{-\log 1.5} = -\frac{1.0792}{0.1761} = -6.128.$$

EXERCISES XIV. E

Solve:

1. $2^x = 20$.
2. $5^x = 3$.
3. $6^x = 135$.
4. $3^x = 0.2540$.
5. $3^{2x} = 350$.
6. $2^{3x} = 15$.
7. $5^{-x} = 10$.
8. $3^{1/x} = 72$.
9. $7^{2x-1} = 3$.
10. $(11)^{1-x} = 0.9232$.

11. $2^{3x+1} \cdot 5^x = 7$.

12. $3^{1-x} \cdot 2^{4x} = 4^x$.

13. $3^{-x} \cdot 5^{3-2x} = 2^{x-4}$.

14. $2 \cdot 3^{2x+7} = 5 \cdot 7^{5x}$.

15. $(1.03)^x = 1.558$.

16. $(1.025)^{-x} = 0.8623$.

17. $\dfrac{(1.02)^x - 1}{0.02} = 62.61$.

18. $\dfrac{1 - (1.015)^{-x}}{0.015} = 0.7649$.

19. $a^{x+b} = c$.

20. $a^x = b^{2x+1} \cdot c^{3-2x}$.

21. $3^x \cdot 2^y = 17$,
$5^x \cdot 7^y = 28$.

22. $5^{x-y} \cdot 3^x = 1$,
$4^{x+y} \cdot (11)^y = 3$.

23. $7^{3x+2y-5} \cdot 3^y = 15$,
$2^{2x-y} \cdot 5^{x+3y} = 36$.

24. $9^{x+y+1} \cdot 2^{2x-y-4} = 100$,
$(12)^{x-5y-2} \cdot 5^{3x-2y+3} = 42$.

25. $x^2 \cdot 3^y = 24$,
$x^3 \cdot 2^y = 18$.

26. $2x^2 \cdot 3^y = 19$,
$4x^3 \cdot 5^{2y} = 7$.

27. $a^x = b^y$,
$a^{x+y} = c$.

28. $a^x = b \cdot c^y$,
$a^{x+b} = b^{y+c}$.

132. Other bases than 10.

In section 123 it was indicated that other numbers than 10 could be used as bases for systems of logarithms. In fact, any positive number except 1 can be used as a base. We recall from the section just referred to that, since $2^3 = 8$, the logarithm of 8 is 3 when 2 is the base; that is, the logarithm is the power to which 2 must be raised to produce 8. In abbreviated form we write

$$\log_2 8 = 3,$$

the subscript indicating the base.

The exercises in the following set can be solved by first rewriting the equations in exponential form and then completing the solution either by inspection or by the method of section 131.

EXERCISES XIV. F

Find the value of x, given that

1. $x = \log_2 32$.
2. $\log_x 9 = 2$.
3. $\log_2 x = -4$.
4. $\log_x 49 = 2$.
5. $\log_5 0.04 = x$.
6. $\log_x 5 = 0.5$.
7. $\log_x 9 = 4$.
8. $x = \log_9 3$.
9. $\log_{\sqrt{2}} x = 7$.

10. $\log_{0.5} x = -6$. **11.** $\log_9 27 = x$. **12.** $\log_{27} x = -\frac{2}{3}$.
13. $\log_x 81 = -4$. **14.** $\log_{\sqrt{2}} 32 = x$. **15.** $\log_x 3.375 = -3$.
16. $\log_{2/3} \frac{16}{81} = x$. **17.** $\log_{25} x = -\frac{1}{2}$. **18.** $\log_x 64 = 4$.
19. $x = \log_{36} \sqrt{6}$. **20.** $\log_x 4 = -\frac{1}{3}$. **21.** $\log_8 x = -\frac{1}{2}$.
22. $\log_b x = 0$. **23.** $\log_x 1 = 0$. **24.** $\log_x a = -2$.
25. $\log_b x = -3$. **26.** $\log_b b = x$. **27.** $\log_b x = b$.
28. $\log_x a = 0$. **29.** $\log_x a = -\frac{1}{2}$. **30.** $\log_b \sqrt{b} = x$.

133. Common and natural logarithms.

Logarithms to the base 10 are called **common, or Briggsian, logarithms.** Their principal advantages are that the characteristic is easily determined, and that the mantissa is the same for all numbers having the same sequence of digits. This is not the case for systems of logarithms employing other bases.

One very important system is that of **natural, or hyperbolic, logarithms** (also called **Napierian logarithms**), which have as a base the number

$$e = 2.71828 \cdots,$$

which can be found to as close an approximation as desired by taking more and more terms of the series

$$1 + \frac{1}{1} + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} + \cdots.$$

(The symbol $\cdots$ is the symbol of **continuation** and may be read "and so on.") The natural logarithm of x may be written $\log_e x$, although the notation **ln** x for $\log_e x$, and lg x for $\log_{10} x$, is sometimes employed.

Natural logarithms are extremely useful, as will be evident to the student who proceeds into the study of calculus. For example, they are needed in finding the length of the path of a projectile, the amount of work done by an expanding gas, the time required for a heated object to cool to a given temperature, the time for a colony of bacteria to increase to a given size. In certain analytic work in advanced mathematics they are indispensable.

134. Change of base.

It is frequently desirable to change from one base to another. This can be done as follows:

Let $x = \log_a N, \qquad y = \log_b N.$

Then, $a^x = N, \qquad b^y = N,$

and $a^x = b^y.$

Take logarithms to the base a:

$$x = y \log_a b,$$

or

$$\log_a N = \log_b N \cdot \log_a b. \tag{1}$$

In particular, if we set $N = a$, we have, since $\log_a a = 1$,

$$1 = \log_b a \cdot \log_a b,$$

or

$$\log_a b = \frac{1}{\log_b a}. \tag{2}$$

Example.

$$\log_2 8 = 3, \qquad \log_8 2 = \frac{1}{3}.$$

To change from a natural logarithm to a common logarithm, we set $a = 10$, $b = e$ in (1), obtaining

$$\log_{10} N = \log_e N \cdot \log_{10} e = \log_e N \cdot \log_{10} 2.7183,$$

or

$$\log_{10} N = 0.4343 \log_e N. \tag{3}$$

Inversely, we have

$$\log_e N = \log_{10} N \cdot \log_e 10 = \log_{10} N \cdot \frac{1}{\log_{10} e},$$

or

$$\log_e N = \frac{\log_{10} N}{0.4343} = 2.3026 \log_{10} N. \tag{4}$$

EXERCISES XIV. G

1. Find $\log_3 75.3$.

SOLUTION. Let $x = \log_3 75.3$. Then $3^x = 75.3$, and taking common logarithms of both sides, we get

$$x \log_{10} 3 = \log_{10} 75.3,$$

or

$$0.4771x = 1.8768,$$

from which it is found that

$$x = 3.934.$$

The same result can be obtained by setting $a = 3$, $N = 75.3$, $b = 10$ in (1).

2. Find $\log_e 75.3$.

SOLUTION. From (4) we find

$$\log_e 75.3 = 2.3026 \log_{10} 75.3$$
$$= 2.3026 \times 1.8768 = 4.3215.$$

Find:

3. $\log_2 5$. **4.** $\log_5 2$. **5.** $\log_e 5$.
6. $\log_5 e$. **7.** $\log_e 2$. **8.** $\log_{100} 1000$.
9. $\log_3 30$. **10.** $\log_5 82$. **11.** $\log_{20} 25$.
12. $\log_{25} 20$. **13.** $\log_{\sqrt{2}} 4.87$. **14.** $\log_5 2.34$.
15. $\log_e 17.3$. **16.** $\log_e 1.73$. **17.** $\log_e 0.173$.
18. $\log_e \pi$. **19.** $\log_e \sqrt{e}$. **20.** $\log_e \pi^e$.

135. Miscellaneous equations.

In section 115 it was stated that the method given there for obtaining irrational roots of integral rational equations could be applied to other types of equations. We shall apply it here to a mixed equation containing both x and $\log x$.

Example.

Find a root of the equation

$$f(x) = \log x - x + 2 = 0.$$

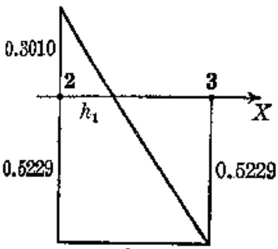


FIG. 35

SOLUTION. First approximation:

$$x_1 = 2.$$

x	$\log x$	$f(x)$
1	0	1
2	0.3010	0.3010
3	0.4771	−0.5229

In Fig. 35,

$$\frac{h_1}{1} = \frac{0.3010}{0.3010 + 0.5229} = \frac{0.3010}{0.8239}, \quad h_1 = 0.4.$$

Second approximation:

$$x_2 = x_1 + h_1 = 2.4.$$

x	$\log x$	$f(x)$
2.3	0.3617	0.0617
2.4	0.3802	−0.0198

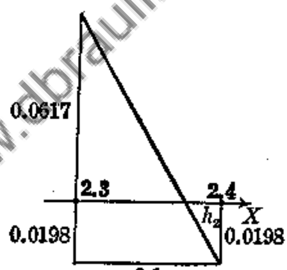


FIG. 36

In Fig. 36,

$$\frac{h_2}{0.1} = \frac{0.0198}{0.0198 + 0.0617} = \frac{0.0198}{0.0815}, \quad h_2 = 0.02.$$

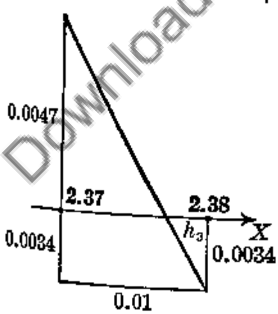


FIG. 37

Third approximation:

$$x_3 = x_2 - h_2 = 2.38.$$

x	$\log x$	$f(x)$
2.37	0.3747	0.0047
2.38	0.3766	−0.0034

In Fig. 37,

$$\frac{h_3}{0.01} = \frac{0.0034}{0.0034 + 0.0047} = \frac{0.0034}{0.0081}, \quad h_3 = 0.0042.$$

Fourth approximation:

$$x_4 = x_3 - h_3 = 2.3758.$$

This is correct to five figures. Four-place tables do not permit further advance.

EXERCISES XIV. H

Solve:

1. $8 \log_{10} x - x = 0.$
2. $2 \log_{10} x + x = 0.$
3. $\log_{10} x = \dfrac{1}{x}.$
4. $\log_{10} x = \dfrac{1}{\sqrt{x}}.$
5. $3^x = 3x.$
6. $3^x = \dfrac{1}{x}.$
7. $e^{-x} = \sqrt{x}.$
8. $\log_{10} x = 2 - x^2.$
9. $2^x = x^2.$
10. $2^{-x} = \log_{10} x.$
11. $\dfrac{(1 + x)^{10} - 1}{x} = 11.$
12. $\dfrac{1 - (1 + x)^{-10}}{x} = 9.$

CHAPTER XV

Compound Interest and Annuities

136. Compound amount.

Interest is money paid for the use of borrowed money. A sum of money placed at interest is called the **principal.** If the interest is added to the principal at the ends of equal intervals of time the interest is said to be **compounded,** or to be **converted** into principal. The interval of time is called the conversion, or interest, **period.** This is usually a year, half-year, or quarter. If no period is stated it will be understood to be a year. The sum to which the principal accumulates is called the **compound amount,** or simply the **amount.** The **rate** is the interest paid on one unit of capital for one period. (In other words, it is the ratio of the interest, for one period, to the principal.) For example, a rate of 3 per cent (i.e., per hundred) means that \$3 per period would be paid on \$100, or \$0.03 per period on \$1. In problems it is best to write the rate as a decimal. In this example the rate would be written 0.03.

If interest is compounded oftener than once a year, it is customary to quote the rate on an annual basis. For example, a rate of 3 per cent compounded semiannually means actually a rate of $1\frac{1}{2}$ per cent for each half-year period. Such a rate is called a **nominal rate** of 3 per cent compounded semiannually.

If P is the principal and r the rate, the interest due at the end of the first period is Pr. The amount at the end of the first period is $P + Pr$, or $P(1 + r)$. It is seen that the amount at the end of one period can be obtained by multiplying the principal P by $(1 + r)$. For example, the amount, at the end of one year, of \$1000 at 3 per cent is \$1000 × 1.03 = \$1030.

The principal for the second period is $P(1+r)$. The amount at the end of the second period is found by multiplying this principal, $P(1+r)$, by $(1+r)$, getting $P(1+r)^2$.

Similarly, the amount at the end of three periods is $P(1+r)^3$.

In general, the amount at the end of n periods is

$$A = P(1+r)^n. \tag{1}$$

For example, the amount of \$1250 at 2 per cent for 3 years is \$1250(1.02)^3.

137. Present value.

The **present value** of a sum of money due at a future date is the principal which, invested at interest, will accumulate to the specified sum at the end of the time. It can be found by solving for P in equation (1) of the preceding section. Thus, the present value of an amount A due in n periods, when money can be invested at rate r is

$$P = \frac{A}{(1+r)^n} = A(1+r)^{-n}. \tag{1}$$

Thus, if the interest rate is 4 per cent, the present value of \$2500 to be paid 3 years hence is \$2500/(1.04)^3.

Problems in compound interest can best be solved by means of tables of compound amounts and present values. When no such tables are available logarithms may be resorted to, although results will not ordinarily be very accurate unless logarithms to a large number of places are employed. Results to any desired degree of accuracy can be obtained by using the binomial formula.

Example 1.

Find the amount of \$1000 at 4% interest compounded annually for 5 years.

SOLUTION BY COMPOUND AMOUNT TABLES.

$$A = P(1 + r)^n = \$1000(1.04)^5.$$

In Table III at the back of the book we look in the column headed 4%, and opposite $n = 5$ we find 1.2167. That is, $(1.04)^5 = 1.2167$. Therefore,

$$A = \$1000 \times 1.2167 = \$1216.70.$$

SOLUTION BY LOGARITHMS.

$$\begin{aligned} A &= \$1000(1.04)^5. \\ \log 1.04 &= 0.0170 \\ & \underline{\times 5} \\ & 0.0850 \\ \log 1000 &= \underline{3} \\ \log A &= 3.0850 \\ A &= \$1216 \text{ (approximately)}. \end{aligned}$$

SOLUTION BY BINOMIAL FORMULA.

$$\begin{aligned} (1.04)^5 &= 1 + 5(0.04) + \frac{5 \cdot 4}{1 \cdot 2}(0.04)^2 + \frac{5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3}(0.04)^3 + \cdots \\ &= 1 + 0.20 + 0.016 + 0.00064 + \cdots = 1.21664, \\ A &= \$1000 \times 1.21664 = \$1216.64. \end{aligned}$$

This is correct to within one cent, the accurate value being \$1216.65 to the nearest cent. One more term of the binomial formula would give this accurate value.

Example 2.

Find the present value of \$1250 due in 3 years if the rate of interest is 4%.

SOLUTION BY PRESENT VALUE TABLES.

$$P = A(1 + r)^{-n} = \$1250(1.04)^{-3}.$$

In the 4% column of Table IV we find, opposite $n = 3$, the value 0.88900. Thus,

$$P = \$1250 \times 0.88900 = \$1111.20.$$

SOLUTION BY LOGARITHMS.

$$P = \frac{\$1250}{(1.04)^3}.$$

$$\begin{aligned} \log 1.04 &= 0.0170 \\ &\underline{\quad\times 3} \\ \log (1.04)^3 &= 0.0510 \end{aligned}$$

$$\begin{aligned} \log 1250 &= 3.0969 \\ \log (1.04)^3 &= \underline{0.0510} \\ \log P &= 3.0459 \\ P &= \$1112. \end{aligned}$$

SOLUTION BY BINOMIAL FORMULA.

$$P = \$1250(1.04)^{-3}.$$

$$(1.04)^{-3} = 1 - 3(0.04) + \frac{-3(-4)}{1 \cdot 2}(0.04)^2 + \frac{-3(-4)(-5)}{1 \cdot 2 \cdot 3}(0.04)^3 + \frac{-3(-4)(-5)(-6)}{1 \cdot 2 \cdot 3 \cdot 4}(0.04)^4 - \cdots$$

$$= 1 - 0.12 + 0.0096 - 0.00064 + 0.00003840 - \cdots$$

$$= 0.8889984.$$

$$P = \$1250 \times 0.8889984 = \$1111.25.$$

EXERCISES XV. A

1. Find the amount of \$100 for 5 years at 4% compounded (a) annually, (b) semiannually, (c) quarterly.
2. Find the present value of \$1000 due in 6 years, the rate of interest being 4% compounded (a) annually, (b) semiannually, (c) quarterly.
3. What is the amount of \$100 at the end of a year if the interest rate is 1% per month?

4. What is the present value of \$100 due in 1 year if the interest rate is 1% per month?
5. How much must be deposited in a savings bank paying $1\frac{1}{2}$% to amount to \$1250 in 5 years?
6. How many years will it take for \$100.00 to amount to \$137.85 at $2\frac{1}{2}$%?
7. In how many years will \$125.00 amount to \$192.43 at 4%?
8. If \$100.00 amounts to \$122.99 in 7 years what is the rate?
9. If \$180.00 amounts to \$374.20 in 15 years what is the rate?
10. A Series E War Savings Bond which costs \$75 is redeemed at the end of 10 years for \$100. What rate of interest does it yield?
11. A Series F War Savings Bond is purchased for \$74 and is redeemed 12 years later for \$100. What rate of interest does it yield?
12. If the present value of \$100.00 due in 6 years is \$86.23 what is the rate?
13. The present value of \$240.00 due in 12 years is \$149.90. Find the rate.
14. How long would it take a sum of money to double itself at (a) 6%? (b) 5%? (c) 4%? (d) 3%? (e) 2%? (f) 1%?
15. How long would it take a sum of money to double itself at 6% compounded (a) annually? (b) semiannually? (c) quarterly? (d) monthly?
16. What rate would cause a sum of money to double itself in (a) 10 years? (b) 12 years? (c) 25 years? (d) 50 years?
17. How long will it take \$250 to amount to \$322 at 3% compounded semiannually?
18. The present value of \$150.00 at 5% compounded semiannually is \$117.18. Find the number of years.
19. The compound amount for a fractional part of a year is defined by the usual formula $S = P(1 + r)^n$, in which fractional values of n are permissible. Find the compound amount of \$100 at 4% (compounded annually) for (a) 6 months, (b) 3 months, (c) 1 month.
20. What single sum of money paid at the end of 4 years will fairly discharge two debts, one of \$2000 due in 3 years and another of \$5000 due in 6 years if the interest rate is 4%?

SUGGESTION. The fair amount would be the compound amount of the \$2000 for 1 year plus the value of the \$5000 2 years before it is due.

21. The population of a city is 200,000 and is increasing at the rate of $2\frac{1}{2}\%$ per year. If it continues to increase at the same rate what will it be in 10 years?

138. Annuity.

A series of equal periodic payments is called an **annuity.** Thus, a series of equal annual payments on a piece of property is an annuity. Likewise, the monthly payments of rent for a house or an apartment constitute an annuity. However, unless otherwise stated, it will be assumed that the payments are annual and that the first payment is made at the *end* of the first year. For example, a three-year annuity of \$1000 would consist of three payments of \$1000 each, one payment made at the end of one year, another at the end of two years, and the last at the end of three years.

139. Amount of an annuity.

We shall derive the formula for the amount of an annuity (that is, the sum to which the periodic payments accumulate at compound interest) on the basis of payments of one dollar each. If each payment is R dollars, instead of one dollar, the accumulation will be R times as much.

If the annuity continues for n years there will of course be n payments. Since the first payment is made at the end of the first year, it will be at interest for $n - 1$ years, and will, by formula (1) of section 136, amount to $(1 + r)^{n-1}$. The second payment will be at interest for $n - 2$ years, and will amount to $(1 + r)^{n-2}$. The next to the last payment will be at interest for one year, and will amount to $(1 + r)$. The final payment of one dollar is made at the end of the n-year term, and its value at that time will consequently be 1. The total amount of the annuity will be the sum of the amounts of the separate payments. Writing these in reverse order, we have for their sum,

$$s = 1 + (1 + r) + \cdots + (1 + r)^{n-2} + (1 + r)^{n-1}. \quad (1)$$

This is a geometric progression of n terms whose common ratio is $(1 + r)$. Its sum, by formula (3) of section 87, we find to be

$$s = \frac{(1 + r)^n - 1}{(1 + r) - 1},$$

or

$$s = \frac{(1 + r)^n - 1}{r}. \quad (2)$$

If each payment is R instead of 1 the amount of the annuity will be Rs, or

$$S = R\,\frac{(1 + r)^n - 1}{r}. \quad (3)$$

The amount of an annuity of \$500 a year for 3 years at $2\frac{1}{2}$ per cent, for example, is

$$\$500 \times \frac{(1.025)^3 - 1}{0.025}.$$

140. Present value of an annuity.

The present value of an annuity is the sum of the present values of all of the payments. The present value of the first payment of an annuity of one dollar is $(1 + r)^{-1}$; the present value of the second payment is $(1 + r)^{-2}$; and so on. The last payment is made at the end of n years, and has a present value of $(1 + r)^{-n}$. The total present value of the annuity is

$$p = (1 + r)^{-1} + (1 + r)^{-2} + \cdots + (1 + r)^{-n}. \quad (1)$$

This is a geometric progression of n terms with common ratio $(1 + r)^{-1}$. By formula (4) of section 87, we find

$$p = (1 + r)^{-1} \cdot \frac{1 - (1 + r)^{-n}}{1 - (1 + r)^{-1}}$$

$$= \frac{1}{1 + r} \cdot \frac{1 - (1 + r)^{-n}}{1 - (1 + r)^{-1}},$$

or

$$p = \frac{1 - (1 + r)^{-n}}{r}. \tag{2}$$

If the annual payment is R instead of 1 the present value of the annuity will be Rp, or

$$P = R\frac{1 - (1 + r)^{-n}}{r}. \tag{3}$$

For example, the present value of an annuity of \$500 a year for 3 years at $2\frac{1}{2}$ per cent is

$$\$500 \times \frac{1 - (1.025)^{-3}}{0.025}.$$

Problems in annuities are best solved by means of annuity tables. When these are not accessible we can use logarithms, or, if greater accuracy is desired, the binomial formula.

Example 1.

Find the amount of an annuity of \$1000 a year for 5 years at 4%.

Solution by Annuity Tables.

$$S = R\frac{(1 + r)^n - 1}{r} = \$1000 \times \frac{(1.04)^5 - 1}{0.04}.$$

In Table V, in the 4% column, we find opposite $n = 5$ the value 5.4163. Thus,

$$S = \$1000 \times 5.4163 = \$5416.30.$$

SOLUTION BY LOGARITHMS.

$$S = \$1000 \times \frac{(1.04)^5 - 1}{0.04}.$$

$$\begin{aligned} \log 1.04 &= 0.0170 \\ &\underline{\times 5} \\ \log (1.04)^5 &= 0.0850 \\ (1.04)^5 &= 1.216. \end{aligned}$$

$$S = \$1000 \times \frac{1.216 - 1}{0.04} = \$5400 \text{ (approximately).}$$

SOLUTION BY BINOMIAL FORMULA.

$$\begin{aligned} (1.04)^5 &= 1 + 5(0.04) + 10(0.04)^2 + 10(0.04)^3 + 5(0.04)^4 \\ &\qquad + (0.04)^5 \\ &= 1 + 0.20 + 0.016 + 0.00064 + 0.00001280 \\ &\qquad + 0.0000001024 \\ &= 1.2166529. \end{aligned}$$

$$S = \$1000 \times \frac{1.2166529 - 1}{0.04} = \$5416.32.$$

Example 2.

Find the present value of an annuity of $1250 a year for 3 years at 4%.

SOLUTION BY ANNUITY TABLES.

$$P = R\frac{1 - (1 + r)^{-n}}{r} = \$1250 \times \frac{1 - (1.04)^{-3}}{0.04}.$$

In Table VI, in the 4% column, we find opposite $n = 3$ the value 2.7751. Thus,

$$P = \$1250 \times 2.7751 = \$3468.90.$$

SOLUTION BY LOGARITHMS.

$$P = \$1250 \times \frac{1 - (1.04)^{-3}}{0.04}.$$

$$\begin{aligned}
\log 1.04 &= 0.0170 \\
&\ \times -3 \\
\log (1.04)^{-3} &= -0.0510 = 0.9490 - 1 \\
(1.04)^{-3} &= 0.8892.
\end{aligned}$$

$$P = \$1250 \times \frac{1 - 0.8892}{0.04} = \$3462 \text{ (approximately)}.$$

SOLUTION BY BINOMIAL FORMULA. In example 2 of section 137 we found by the binomial formula $(1.04)^{-3} = 0.8889984$.

$$P = \$1250 \times \frac{1 - 0.8889984}{0.04} = \$3468.80.$$

The correct value (found by using annuity tables to 6 decimal places) is \$3468.86. In order to obtain this value, the binomial expansion used in calculating $(1.04)^{-3}$ should be carried to one more term.

EXERCISES XV. B

1. Find the amount of an annuity of \$1000 a year for 5 years at 4%.
2. Find the present value of an annuity of \$500 a year for 3 years at 4%.
3. A man puts \$200 into a certain fund at the end of each year. If the fund earns 3% interest, to how much has it accumulated at the end of 10 years?
4. The purchaser of a piece of property pays \$2000 down and agrees to pay \$1000 at the end of each year for 12 years. What is the equivalent cash value of the property if the interest rate is 5%?
5. A man buys a house, paying \$1500 down and \$500 at the end of each year for 15 years. Find the equivalent cash payment, assuming an interest rate of 6%.

6. A man has a debt of \$5000 due in 3 years. In order to pay it off he deposits a certain amount, at the end of each year for the 3 years, in a fund bearing 3% interest. How much must his annual payment be? (A fund of this sort is called a **sinking fund.**)
7. A man buys a house for \$8000. He pays \$1000 down and agrees to pay the balance, including principal, and interest at 5%, in 10 equal installments. What annual payment must he make?
8. A piece of property is purchased for \$15,000. The purchaser makes a down payment of \$3000 and agrees to pay the balance, including principal, and interest at 4%, in 8 equal annual installments. Find the amount of each installment.
9. A quarry yields a net annual income of \$10,000. It is estimated that the rock will be exhausted in 15 years. What price could a capitalist afford to pay for the quarry if he is to make 6% on his investment?
10. A man deposits \$500 at the end of each 6 months in a bank paying 2% compounded semiannually. How much will he have to his credit at the end of 8 years, assuming that he makes no withdrawals?
11. Find the present value of the annuity of the preceding exercise.
12. A bond has attached to it 10 coupons. At the end of each year a coupon is detached by the owner of the bond and is cashed for \$2.50. At the end of 10 years he turns in the bond and receives \$100.00. What is the present value of the bond (including the coupons) if the current rate of interest is 4%?
13. A manufacturing company has a machine which will be completely worn out in 6 years. How much must be set aside at the end of each year, in a fund invested at 4%, to have a sufficient amount to purchase a new machine costing \$3000 when the old one is worn out? (A fund of this sort is called a **depreciation fund.**)
14. A son receives, by the terms of his father's will, an annuity of \$2500 a year for 20 years, beginning at the end of 1 year. The state imposes an inheritance tax of 1%. What is the amount of the tax if money is worth 3% (i.e., if the current interest rate is 3%)?

15. A man puts \$500 a year, at the beginning of each year for 10 years, into a fund which is invested at 4%. What will be the amount in the fund at the end of the 10 years?

16. The beneficiary of a life insurance policy is to receive \$1000 a year for 10 years, the first payment to be made at the time of death of the insured. Find the value of this annuity at the time of death of the insured, assuming the current rate of interest to be 4%.

17. A man puts \$1000 at the end of each year for 6 years, into a fund which is invested at 4%. If the fund is allowed to accumulate for 6 years more, without any additional payments being made, what will be its amount at the end of the 12-year period?

18. A man invests \$6000 at 2% at the time his son is born. The boy is to receive the money in 6 equal installments, the first of which is to be paid on his 21st birthday. Find the amount of each installment.

19. A man wishes to provide, at the time of the birth of his son, for the boy's college education. If the boy is to receive \$2000 a year for 4 years, the first payment to be made on his 18th birthday, how much must the father set aside, assuming that the money can be invested at 2%?

20. A student borrows \$400 at the beginning of each of his 4 years in college, signing a note to pay off the debt, with interest at 4%, in 5 equal installments, the first installment to be paid 5 years after the first \$400 was borrowed. Find the amount of each installment.

Solve the next two exercises by the method of section 115 or that of section 117:

21. An annuity of \$500.00 a year for 4 years amounts to \$2128.50. What is the rate?

22. The present value of an annuity of \$300.00 a year for 3 years is \$837.60. Find the rate.

CHAPTER XVI

Permutations and Combinations

141. Fundamental principle.

If one thing can be done in m different ways, and if after it has been done in any of these ways a second thing can be done in n different ways, the two things can be done in the order stated in mn different ways.

This principle can obviously be extended to the case in which more than two things are involved.

Example 1.

How many three-place numbers can be formed with the digits 1, 2, 3, 5 if each digit can be used only once?

Solution. The hundreds' place can be filled in any one of 4 different ways, the tens' place can then be filled in any one of 3 different ways, finally the units' place can be filled in 2 different ways. Thus, we can form $4 \times 3 \times 2 = 24$ numbers.

Example 2.

How many three-place numbers can be formed with the digits 1, 2, 3, 5 if any digit can be repeated?

Solution. The hundreds' place can be filled in 4 ways, the tens' place can then be filled in 4 ways, and the units' place can also be filled in 4 ways. We can form $4 \times 4 \times 4 = 64$ numbers.

Example 3.

How many three-place numbers less than 300 can be formed with the digits 1, 2, 3, 5 if repetition of digits is not allowed?

Solution. The hundreds' place can be filled in 2 different ways (with 1 or 2 only, since 3 or 5 in this place would make the number greater than 300). The tens' place can then be filled in 3 ways and the units' in 2. Thus, $2 \times 3 \times 2 = 12$ numbers can be formed.

Example 4.

How many even numbers of three places can be formed with the digits 1, 2, 3, 5, repetition of digits not allowed?

Solution. The units' place can be filled in only 1 way (with the 2). The tens' place can then be filled in 3 ways, and the hundreds' in 2. There are $1 \times 3 \times 2 = 6$ numbers.

142. Permutations.

Each arrangement in order of a set of things is called a **permutation** of the set. Thus, the set of letters a, b, c, if we use all of them, can be arranged in the following orders:

abc	*bac*	*cab*
acb	*bca*	*cba*

We see that there are six permutations. This can be determined by noting that the first place can be filled by any one of the three letters, that is, in 3 different ways; after the first place is filled the second place can be filled by either of the two remaining letters. The first two places can therefore, according to our fundamental principle, be filled in $3 \times 2 = 6$ ways. After the first two places are filled the letter to go into the third place is determined. Thus, the three letters can be permuted in $3 \times 2 \times 1 = 3! = 6$ ways.

If we have four letters, a, b, c, d, and take them two at a time, we can form the permutations:

ab	*ba*	*ca*	*da*
ac	*bc*	*cb*	*db*
ad	*bd*	*cd*	*dc*

We note that the first place can be filled in 4 different ways, the second in 3, and consequently the two places in $4 \times 3 = 12$ different ways.

In general, if we have n different things and take r of them at a time, we can fill the first place in n different ways. After the first place is filled there are $n - 1$ things left, and any one of these can be put into the second place. That is, the second place can be filled in $n - 1$ different ways. The first two places can therefore be filled in $n(n - 1)$ different ways. The third place can be filled in $n - 2$ different ways, and so on to the rth place, which can be filled in $n - r + 1$ different ways. Thus, the number of permutations of n things taken r at a time is given by the formula

$$ {}_nP_r = n(n-1)\cdots \text{to } r \text{ factors} \tag{1}$$
$$ = n(n-1)\cdots(n-r+1). \tag{2}$$

If we set $r = n$, we find that the number of permutations of n things taken all at a time is

$$ {}_nP_n = n(n-1)\cdots 3\cdot 2\cdot 1 = n\,!. \tag{3}$$

143. Permutations of things some of which are alike.

Suppose that we have five letters a, a, a, b, c. Consider any permutation of them such as $abaac$. If the a's were different, say a_1, a_2, a_3, we should have 3 ! permutations similar to this, namely,

$a_1ba_2a_3c$	$a_2ba_1a_3c$	$a_3ba_1a_2c$
$a_1ba_3a_2c$	$a_2ba_3a_1c$	$a_3ba_2a_1c$

These permutations would be indistinguishable if the subscripts were removed. Thus, to any given permutation such as $abaac$, in which the a's are alike, there corresponds a set of 3 ! permutations in which the a's are different. That is, there are 3 ! times as many permutations when the a's are different as there are when the a's are alike. But for five different letters a_1, a_2, a_3, b, c there are 5 ! permutations. Consequently, if the three a's are alike the number of permutations is $5! \div 3! = 20$.

In general, if there are n things of which n_1 are alike of one kind, n_2 alike of another kind, and so on, the number of permutations of these things taken all at a time is

$$P = \frac{n!}{n_1!\,n_2!\cdots}.$$

This can be established by the same line of reasoning employed for the special case.

EXERCISES XVI. A

1. In how many different orders can 6 people be seated in a row?
2. In how many different orders can 6 people be seated at a round table?

   Suggestion. Consider the position of one of the persons as fixed.

3. A theater party is composed of 3 men and 3 women. They have 6 seats in a row. In how many different orders can they be seated so that no 2 persons of the same sex will be together?
4. After the theater, the members of the party of the preceding exercise visit a night club. In how many different orders can they be seated at a round table so that no 2 persons of the same sex will be next to each other?
5. In how many different orders can 3 married couples be seated in a row so that no 2 persons of the same sex will be together and so that no man will be next to his wife?
6. In how many different orders can 3 married couples be seated about a round table so that no 2 persons of the same sex will be together and so that no man will be next to his wife?
7. Five persons took a trip of 240 miles in a 5-seated automobile. Before starting out they decided that in order to break the monotony of the trip they would change about from time to time until all possible seating arrangements had been used. They agreed that a good plan would be to travel the same

distance under each seating arrangement. If this plan were followed how frequently would they have to change?

8. An airplane has 6 seats. In how many different orders can 6 people be placed in these seats when the plane is in flight if only 2 of them can occupy the pilot's seat?

9. How many different automobile license numbers of 6 places or fewer are possible if no number can begin with a zero?

10. How many different automobile license numbers can be made by using 5 or fewer digits preceded by a letter if the digit immediately following the letter cannot be a zero and the letters O and I are excluded?

11. How many different 4-place odd numbers can be formed with the digits 1, 2, 3, 4, 5, 6 (a) if repetitions are not allowed? (b) if repetitions are allowed?

12. How many different numbers less than 500 can be formed with the digits 1, 2, 3, 4, 5, 6 (a) if repetitions are not allowed? (b) if repetitions are allowed?

13. How many different numbers greater than 500 can be formed with the digits 1, 2, 3, 4, 5, 6 if repetitions are not allowed?

14. How many different 4-place numbers can be formed with the digits 0, 1, 2, 3, 4, 5 (a) if repetitions are not allowed? (b) if repetitions are allowed?

15. A shelf contains 7 books bound in red and 4 bound in green. In how many different orders can they be arranged (a) if books of the same color must be kept together? (b) if the green books must be kept together but the red books may be separated?

16. There are 4 positions on a flagstaff and 6 different colors of flags (at least 4 flags of each color). How many different signals can be made by displaying 4 flags simultaneously?

17. There are 7 positions on a flagstaff and 7 flags, of which 3 are red, 2 white, and 2 blue. How many different signals can be made by displaying all of the flags simultaneously?

18. How many different permutations can be made of the letters of the following words? (a) "degree," (b) "abscissa," (c) "independent."

19. Six persons wish to select 1 book each from a shelf containing 12 books, of which 8 are in English and 4 in Spanish. How many different selections are possible (a) if 1 of the persons

cannot read Spanish and must select a book in English? (b) if 3 of the persons cannot read Spanish and must select a book in English? (c) if 2 of the persons must select a book in English and 1 must select a book in Spanish?

20. A station wagon has a seating capacity for 8 persons: 2 besides the driver in the front seat, 2 in the middle seat, and 3 in the rear seat. In how many different orders can 8 persons be seated if only 3 can drive and 1 of the others refuses to occupy a rear seat?

144. Combinations.

A set of things without reference to the order in which they are arranged is called a **combination.** Thus, *abc*, *acb*, *bac*, *cab*, are the same combination, although they are different permutations. A formula for the number of combinations of n different things taken r at a time can readily be derived from the formula for the number of permutations of n things taken r at a time. For, corresponding to each combination of the r things there are $r\,!$ permutations. Therefore, if ${}_nC_r$ denotes the number of combinations of n things taken r at a time,

$$ {}_nP_r = r\,!\,{}_nC_r, \tag{1} $$

from which we get

$$ {}_nC_r = \frac{{}_nP_r}{r\,!}, \tag{2} $$

or

$$ {}_nC_r = \frac{n(n-1)\cdots \text{to } r \text{ factors}}{1 \cdot 2 \cdots r} \tag{3} $$

$$ = \frac{n(n-1)\cdots(n-r+1)}{r\,!}. \tag{4} $$

If we multiply numerator and denominator by $(n-r)\,!$ we reduce (4) to the form

$$ {}_nC_r = \frac{n\,!}{r\,!(n-r)\,!}. \tag{5} $$

We note that the formula (5) is also the formula for ${}_nC_{n-r}$. That is,

$$ {}_nC_r = {}_nC_{n-r}, \qquad (6) $$

or in words: *The number of combinations of n things taken r at a time is the same as the number of combinations of n things taken n − r at a time.*

This can also be established by the following reasoning: To each combination of r things which we *take from* the n things, there corresponds a combination of $n - r$ things which we *leave*.

145. Binomial coefficients.

By referring to the binomial formula, (2) of section 77, we see that the coefficient of term number $r + 1$ is (4) of the preceding section, namely, ${}_nC_r$. The binomial formula may therefore be written

$$ (a + b)^n = a^n + {}_nC_1a^{n-1}b + {}_nC_2a^{n-2}b^2 + \cdots + {}_nC_ra^{n-r}b^r + \cdots + {}_nC_nb^n. $$

This can readily be shown as follows:

$$ (a + b)^n = (a + b)(a + b) \cdots (a + b), \text{ } n \text{ factors.} $$

Each term of the expanded product is the product of n letters, one from each of the n factors. Thus, every term involving a^rb^{n-r} is obtained by taking a from any r factors and b from the remaining $n - r$ factors. Therefore, the number of terms involving a^rb^{n-r} is the number of ways in which r things can be selected from n, namely ${}_nC_r$.

146. Total number of combinations.

If we set $a = b = 1$ in the last equation, we get

$$ (1 + 1)^n = 2^n = 1 + {}_nC_1 + {}_nC_2 + \cdots + {}_nC_n, \qquad (1) $$

from which we find

$$_nC_1 + {}_nC_2 + \cdots + {}_nC_n = 2^n - 1, \tag{2}$$

which gives the total number of combinations of n things taken 1, 2, $\cdots$, $n - 1$, or n at a time. For example, the total number of combinations of the three letters a, b, c, taken 1, 2, or 3 at a time, is $2^3 - 1 = 7$, as can readily be verified.

EXERCISES XVI. B

1. How many combinations can be made of the letters a, b, c, d, e, f, g taken 3 at a time?
2. In how many ways can a subcommittee of 3 be chosen from a committee of 9 persons?
3. A captain wishes to select 2 noncommissioned officers to be placed in command of a detail of soldiers. In how many ways can he do this if he has 8 noncommissioned officers to pick from?
4. A student wishes to use 4 different colors to make a map. In how many ways can he do this if he has a selection of 8 colors?

   NOTE. It has been stated that 4 different colors are sufficient to use in making a map so that no countries which have a common boundary will be the same color. A proof of this simple statement has never been effected. It has been proved, however, that 5 colors are always sufficient.

5. A shelf of reserved reference books contains 20 books. If a student is allowed to take out 2 books overnight how many selections can he make?
6. How many different basketball teams of 5 players each can be formed from a squad of 12 men, assuming that any member of the squad can play in any position?
7. How many different kinds of bouquets of 3 kinds of flowers each can be made from 10 varieties of flowers?
8. A committee of 3 seniors and 2 juniors is to be selected from a

group of 8 seniors and 16 juniors. How many different committees can be formed?

9. A committee of 5 is to be chosen from 10 seniors and 15 juniors. If there must be at least 3 seniors on the committee how many different committees can be formed?

10. A box contains 6 black balls and 9 white balls. How many different combinations of 6 balls, of which 2 are black and 4 white, can be selected from the box?

11. How many different combinations of 6 cards each can be dealt from an ordinary deck of 52 playing cards?

12. How many different sums of money can be formed with a penny, a nickel, a dime, a quarter, a half dollar, and a dollar?

13. How many different combinations can be formed from the letters a, b, c, d, e, f, g, taken 1 or more at a time?

14. A publishing company issues a set of bilingual dictionaries covering English, Spanish, Portuguese, French, Italian, and German. Each volume is a 2-way dictionary (e.g., Spanish-French and French-Spanish are in a single volume). How many volumes are there in the set?

15. The tennis squad of one university consists of 10 players, that of another consists of 7 players. In how many ways can a doubles match between the two institutions be arranged?

16. From a list of 10 exercises on permutations and 12 exercises on combinations how many different examinations can be made which will include 5 questions on each of these topics?

17. How many different combinations consisting of 7 black balls, 4 white balls, and 3 red balls can be chosen from a box containing 10 black, 9 white, and 6 red balls?

18. In how many ways can one make a selection of 4 novels, 2 biographies, and 3 books of poems from a shelf containing 10 novels, 7 biographies, and 6 books of poems?

19. In how many ways can one make a selection of 7 books from the shelf in the preceding exercise if he chooses at least 3 novels and at least 2 biographies?

20. (a) Prove that ${}_nC_r + {}_nC_{r-1} = {}_{n+1}C_r$. (Note that Pascal's triangle, section 77, exemplifies this formula.) (b) Use the formula to find the number of committees, consisting of either 3 or 4 members, that can be appointed from a group of 12 persons.

MISCELLANEOUS EXERCISES XVI. C

1. Given ${}_nP_2 = 132$; find n.
2. Given ${}_nC_2 = 105$; find n.
3. In how many different orders can 6 books be arranged on a shelf?
4. In how many different orders can a theater party of 6 persons be seated in a row of 8 seats?
5. In how many different orders can a theater party of 8 persons be seated in a row of 8 seats?
6. How many combinations of 3 different kinds of fruit can be made from apples, bananas, oranges, peaches, and pears?
7. How many combinations of 4 letters each can be made with the letters a, b, c, d, e, f?
8. In how many different orders can 7 children be arranged to dance around a Maypole?
9. An ice cream manufacturer makes 7 varieties of ice cream. How many different combinations of 2 or 3 varieties each are available for making quart bricks?
10. How many different basketball teams (5 players) can be formed from a squad of 10 men if only 2 of the men can play center and these two can play in no other position?
11. In how many ways can a crew of 8 men be seated in their racing shell if only 2 of the men can pull the stroke oar?
12. How many distinct lines can be drawn through 12 points no 3 of which are in the same straight line?
13. How many different planes can be passed through 12 points no 4 of which are in the same plane?
14. A committee of 5 is to be chosen from 9 men and 7 women. In how many ways can the committee be formed if it is to have a majority of men?
15. In how many different orders can the letters of the word "parallel" be arranged?
16. In how many different ways can the letters of the word "formula" be arranged without changing the order of the vowels?
17. In how many different orders can the letters of the word "decimal" be arranged if the first place is to be filled with a consonant and no consonants are to be adjacent to each other?

18. In an 8-team league every team plays each of the others 44 times. How many games are played?
19. How many different baseball nines can be formed from a squad of 20 men if only 5 can pitch, only 3 can catch, and these 8 men can play in no other position?
20. How many different batting orders are possible for the 9 men of a baseball team?
21. How many different numbers can be made of the odd digits if repetitions are not allowed?
22. In how many different ways can 8 books be arranged on a shelf (a) if a certain pair of the books must be together? (b) if a certain pair of the books must not be together?
23. From 7 pairs of gloves, in how many ways can a right-hand and a left-hand glove be selected without taking a pair?
24. Given 6 flags of different colors, how many different signals can be made by displaying them in a vertical line, using any number at a time?
25. Given 6 different colors of flags (at least 6 flags of each color), how many different signals can be made by displaying them in a vertical line, using any number of flags from 1 to 6 at a time?
26. How many different combinations of colors can be made from red, blue, yellow, orange, purple, green, black, and white by using 1 or more at a time?
27. A basketball coach wishes to divide his squad of 10 men into 2 teams for a practice game. In how many ways can he effect the division?
28. How many different 4-place numbers can be written by using 2 odd digits and 2 even digits if 0 is excluded but repetition of digits is allowed?

SOLUTION. There are 5 odd digits (1, 3, 5, 7, 9) and 4 even digits (2, 4, 6, 8). We have four classes of numbers:

(a) Those in which the odd digit is repeated and the even digit is repeated (e.g., 1122).
(b) Those in which the odd digit is repeated but the even digit is not (e.g., 1124).
(c) Those in which the odd digit is not repeated but the even digit is (e.g., 1322).

(d) Those in which neither digit is repeated (e.g., 1324).

We consider each class separately.

(a) The odd digit may be any one of the five and the even digit may be any one of the four. Thus, the number of ways in which the two digits may be selected is $5 \times 4 = 20$. But, according to section 143, the number of ways in which each selection can be arranged is

$$\frac{4!}{2!\,2!} = 6.$$

Therefore, the total number of arrangements in this class is $20 \times 6 = 120$.

(b) The odd digit may be any one of the five and the number of ways in which the even digits may be selected is ${}_4C_2 = 6$. The number of ways in which the selection of the two digits may be made is $5 \times 6 = 30$. Using the formula of section 143 again, we find that the number of ways in which each selection can be arranged is 12. The total number of arrangements in this class is $30 \times 12 = 360$.

(c) The number of ways in which the odd digits may be selected is ${}_5C_2 = 10$. The even digit may be any one of the four. The number of selections is $10 \times 4 = 40$. As in (b), each of these can be arranged in 12 ways, so that the total number of arrangements in this case is $10 \times 40 = 400$.

(d) The number of selections is ${}_5C_2 \times {}_4C_2 = 10 \times 6 = 60$. Each selection can be arranged in $4! = 24$ ways. Consequently, the total number of arrangements in this case is $60 \times 24 = 1440$.

The number of 4-place numbers is, therefore, $120 + 360 + 400 + 1440 = 2320$.

29. How many different arrangements of 6 letters each, consisting of 3 consonants and 3 vowels, can be made from the letters *a, b, c, d, e, f*, if repetitions are allowed?

30. In how many ways can a selection of 3 letters be made from the letters *a, b, c, d, e, f*, if repetitions are allowed?

31. (a) How many selections of 4 letters each can be made from the letters of the word "proportion"? (b) How many arrangements of 4 letters each can be made?

CHAPTER XVII
Probability

147. Probability.

If the ways in which an event can either happen or fail are all equally likely, the **probability** that it will happen in a given trial is the ratio of the number of ways in which it can happen to the total number of ways in which it can either happen or fail. For example, in drawing a card from an ordinary deck, the probability of obtaining an ace is $4/52 = 1/13$, since there are four aces out of a total of fifty-two cards.

In general, if the event being tried can happen in h ways and fail in f ways (all equally likely), the probability that it will happen is given by

$$p = \frac{h}{h + f},$$

and the probability that it will fail is given by

$$q = \frac{f}{h + f}.$$

It is readily seen that $p + q = 1$. Thus, in the card-drawing example cited above, the probability of failing to draw an ace is $\frac{48}{52} = \frac{12}{13} = 1 - \frac{1}{13}$.

It is frequently impossible to analyze an event into the number of equally likely ways in which it can happen or fail. In such cases, however, it may be possible to observe a large number of trials of the event and to record the number of happenings. The **relative frequency** of occurrence of the event is the ratio of the number of happenings to the number of trials. If the number of trials has been large, we assume that this relative frequency is approximately the same as the probability and may be substituted for it.

For example, suppose we had a deck containing an unknown number of cards and an unknown number of aces. If 1000 draws (with replacement of the card drawn before the next draw was made) have resulted in 96 aces we should say that the probability of drawing an ace (as determined by statistical experiment) is 96/1000 or 0.096. Here it would be possible to check our experimental result with the mathematical probability by counting the number of cards and the number of aces.

Suppose, however, that a life insurance company has records on 100,000 men 47 years old, showing that 98,800 of them are alive one year later, the other 1200 having died. We say that the probability that a 47-year-old man will survive at least one year is $98{,}800/100{,}000 = 0.98800$ and the probability that he will die during the year is $1 - 0.98800 = 0.01200$, although it is obviously impossible to analyze the situation into a number of equally likely ways of living or dying.

A table exhibiting the number of persons dying each year out of an initial group of specified size is called a **mortality table.** Such a table is the **American Experience Table of Mortality** (Table VII at end of book).

Example 1.

What is the probability that a person 10 years old will die during the year?

SOLUTION. In the American Experience Table the number living at age 10 is 100,000, the number dying during the year is 749. The probability of dying is $749/100{,}000 = 0.00749$.

Example 2.

Find the probability that a person 25 years old will die during the year.

SOLUTION. The number living at age 25 is 89,032, the number dying during the year is 718. The probability of dying during the year is $718/89032 = 0.008$.

Example 3.

Find the probability that a person 25 years old will live for at least one year.

SOLUTION. The number living at age 25 is 89,032, the number living one year later is 88,314. The probability that a person 25 will live to be 26 is $88314/89032 = 0.992$.

Note that the sum of the probabilities of example 2 and example 3 is 1.

148. Expectation.

If p is the probability of success in a single trial of an event, the **expected number** of successes in n trials is np. The expected number of hearts in making 100 draws from an ordinary deck of cards (replacing the card drawn before making the next draw) is $100 \times \frac{1}{4} = 25$, since the probability of obtaining a heart on a single draw is $\frac{13}{52} = \frac{1}{4}$.

If p is the probability that a person will win an amount of money a, his **expectation** is defined to be pa. It is the average amount that he would expect to win in the long run. For example, if a person is to win \$1.00 provided he is successful in drawing a heart from a deck of cards, his expectation is $\frac{1}{4} \times \$1.00 = \$0.25$.

EXERCISES XVII. A

1. A box contains 6 white balls and 4 black balls, identical except for color. If a random draw is made, what is the probability that the ball is (a) white? (b) black?
2. What is the probability of drawing a red ace from an ordinary deck of 52 playing cards?
3. What is the probability of throwing an even number other than a "6" with a cubical die?
4. The rim of a circular wheel is divided into 27 equal parts, which are marked with the numbers 1 to 27 inclusive. A

fixed pointer at the side indicates one of the numbers when the wheel is at rest. If the wheel is spun what is the probability that, when it stops, the pointer will indicate (a) the number 7? (b) an odd number? (c) a number consisting of 2 digits? (d) a number less than 7? (e) a number divisible by 5? (f) a number divisible by 3?

5. What is the expectation of a person who is to get 25 cents if he throws an even number with a die?

6. A box contains 12 white, 5 red, and 3 black balls. What is the expectation of a person who is to receive a dollar if he draws a red ball?

7. The prize in a lottery, for which 200 tickets are out, is $25. What is the expectation of a person who holds 5 tickets?

8. In a box are 25 envelopes which are indistinguishable from each other in appearance. Each of 3 of them, however, contains a $2 bill. What is the expectation of a person drawing an envelope from the box?

9. (a) What is the probability that a person 20 years old will die within one year? (b) What is the probability that he will live at least one year?

10. (a) What is the probability that a person 90 years old will die within one year? (b) What is the probability that he will live at least one year?

11. Find the probability that a person 25 years old will live at least 5 years longer.

Solution. The American Experience Table shows that 89,032 persons (out of an original group of 100,000) are living at age 25. The number living 5 years later (that is, at age 30) is 85,441. The probability that a person 25 years old will live at least 5 years longer is $85441/89032 = 0.9597$.

12. Find the probability that a person 25 years old will die within 5 years.

Solution. The number dying between ages 25 and 30 is $89{,}032 - 85{,}441 = 3591$. The probability of dying is $3591/89032 = 0.0403$.

13. Find the probability that a person 50 years old (a) will live at least 10 years, (b) will die within 10 years.

14. Find the probability that a person 35 years old (a) will live at least 10 years, (b) will die within 10 years.

15. What is the probability that a person 25 years old will die within the ages of 40 to 42 inclusive?

16. A box contains 6 white balls and 4 black balls. Three balls are taken at random from the box. (a) What is the probability that all are white? (b) What is the probability that 2 are white and 1 black?

SOLUTION. (a) The number of ways that 3 white balls can be drawn from the 6 is ${}_6C_3 = \frac{6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3} = 20$. This is the number of ways in which the event of drawing 3 white balls can happen. The total number of ways in which the event can happen or fail is the number of ways in which 3 balls can be selected from the 10 in the box, viz., ${}_{10}C_3 = \frac{10 \cdot 9 \cdot 8}{1 \cdot 3 \cdot 3} = 120$.

The required probability is $\frac{{}_6C_3}{{}_{10}C_3} = \frac{20}{120} = \frac{1}{6}$.

(b) The number of ways of getting 2 white balls is ${}_6C_2 = \frac{6 \cdot 5}{1 \cdot 2} = 15$. The number of ways of getting 1 black ball is ${}_4C_1 = 4$. The total number of ways of getting 2 white balls and 1 black ball is the product of these two, and the required probability is

$$\frac{{}_6C_2 \cdot {}_4C_1}{{}_{10}C_3} = \frac{15 \cdot 4}{120} = \frac{1}{2}.$$

17. If a box contains 12 white balls and 6 black balls, what is the probability that 2 balls drawn from the box at random (a) will both be white? (b) will both be black? (c) will be of different colors?

18. What is the probability that of 4 cards dealt from a well-shuffled deck 3 will be diamonds and 1 a spade?

19. There are 12 seats in a row at a theater. What is the probability that if 4 seats are taken at random they will all be together?

20. Each of the numbers from 1 to 10 inclusive is written on a separate ticket. These tickets are thoroughly mixed and 2 of

them are drawn. Find the probability that their sum will be 7.

21. Each of the numbers from 1 to 25 inclusive is written on a separate ticket. The tickets are thoroughly mixed and 5 of them are drawn. What is the probability that 2 of them will be odd?

22. Certain manufactured articles are sold in boxes of 50 each. At the factory they are inspected by taking a sample of 5 articles from each box and passing the box as satisfactory if no defective articles are found in the sample. Find the probability that a box will be passed if it contains (a) 1 defective article, (b) 2 defective articles, (c) 3 defective articles.

23. Five red books and 4 blue books are placed at random on a shelf. What is the probability that the blue books will all be together?

24. Four red books, 3 blue books, and 2 green books are placed at random on a shelf. What is the probability that the blue books will all be together and the green books all together?

149. Mutually exclusive events.

Two or more events are **mutually exclusive** if not more than one of them can happen in a given trial. Thus, the drawing of an ace and the drawing of a king (in the same draw of a single card) from a deck of cards are mutually exclusive events. The drawing of an ace and the drawing of a heart are not.

The probability that some one or other of a set of mutually exclusive events will happen in a single trial is the sum of their separate probabilities of happening.

Let us for simplicity consider the case of two mutually exclusive events. Suppose that the first can happen in m_1 ways, the second in m_2 ways, and that the total number of ways in which either of the two events can happen or fail is n. Then the probabilities of happening are $p_1 = m_1/n$ and $p_2 = m_2/n$ respectively.

The m_1 ways in which the first event can happen and the m_2 ways in which the second can happen are mutually exclusive. Consequently the probability that either the first or the second event will happen is

$$\frac{m_1 + m_2}{n} = \frac{m_1}{n} + \frac{m_2}{n} = p_1 + p_2.$$

As an illustration let us calculate the probability of drawing an ace or a king from a deck of cards. According to the theorem this probability is $\frac{1}{13} + \frac{1}{13} = \frac{2}{13}$. As a verification we note that since there are 4 aces and 4 kings, the probability of drawing either an ace or a king is $\frac{4+4}{52} = \frac{8}{52} = \frac{2}{13}$.

It is not difficult to extend the reasoning so as to cover the case of more than two events.

150. Independent events.

The events of a set are **independent** if the happening of any one of them does not affect the happening of the others.

The probability that all of a set of independent events will happen in a trial in which all are possible is the product of their separate probabilities of happening.

For the case of two independent events, suppose that the first can happen in m_1 out of n_1 possible ways of happening or failing, and that the second can happen in m_2 out of n_2 possible ways. The probabilities of happening are p_1 and p_2 respectively. By the fundamental principle of section 141, the two events can happen together in m_1m_2 ways, and the total number of possible ways in which they can happen or fail is n_1n_2. Thus, the probability that both will happen is

$$\frac{m_1m_2}{n_1n_2} = \frac{m_1}{n_1} \cdot \frac{m_2}{n_2} = p_1p_2.$$

The extension to more than two events is obvious.

As an illustration, let us consider the probability of throwing two aces with a pair of dice. According to the theorem the probability is $\frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$. This is verified if we note that there is only one way to obtain two aces (both dice must show aces). There are however $6 \cdot 6 = 36$ possibilities, since any one of the six faces of either die may occur with any one of the six faces of the other. Thus the probability is $\frac{1}{36}$.

151. Dependent events.

If the happening of one event affects the probability of occurrence of another, the second event is **dependent** on the first. For example, in drawing two cards from a deck the probability of obtaining an ace on the second draw depends on whether an ace has been drawn the first time.

If the probability that an event will happen is p_1, and if after it has happened the probability that a second event will happen is p_2, the probability that the first event will happen and will be followed by the second event is p_1p_2.

Suppose that the first event can happen in m_1 out of n_1 possible ways of happening or failing; then $p_1 = m_1/n_1$. Suppose further that after the first event has occurred the second can happen in m_2 ways and happen or fail in n_2 ways; then $p_2 = m_2/n_2$. By the fundamental principle of section 141, the first event and then the second can happen in m_1m_2 ways; an occurrence or failure of the first, followed by an occurrence or a failure of the second, can take place in n_1n_2 ways. Thus, the probability that the first event will happen and be followed by the second is

$$\frac{m_1m_2}{n_1n_2} = \frac{m_1}{n_1} \cdot \frac{m_2}{n_2} = p_1p_2.$$

This proof admits of an easy extension to more than two dependent events.

Example.

Find the probability of obtaining an ace on both the first and second draws from a deck of cards when the first card is not replaced before the second is drawn.

SOLUTION. The probability of obtaining an ace on the first draw is $p_1 = \frac{4}{52} = \frac{1}{13}$. If the first card drawn is an ace there are only 3 aces remaining in the deck, which now consists of 51 cards. Thus, the probability of getting an ace on the second draw is $p_2 = \frac{3}{51} = \frac{1}{17}$. The required probability is $p_1p_2 = \frac{1}{13} \cdot \frac{1}{17} = \frac{1}{221}$.

EXERCISES XVII. B

1. A box contains 10 white balls, 3 black balls, and 2 red balls. What is the probability of drawing a white ball or a red ball in a single random draw?
2. What is the probability of drawing either a red king or the ace of spades in a single draw from an ordinary deck of cards?
3. Each of the 9 letters of the word "seventeen" is written on a separate slip of paper. These slips, which are exactly alike, are placed in a box and thoroughly mixed. What is the probability of obtaining either an *e* or an *n* in a single draw?
4. Each of the letters *a*, *b*, *c*, *d*, *e* and each of the numbers from 1 to 15 inclusive is written on a separate slip of paper. These slips, which are identical, are placed in a box and thoroughly mixed. What is the probability, in a single draw, of obtaining either a vowel or an even number?
5. Find the probability that a person 20 years old will die either between the ages 40–49 inclusive or between the ages 60–69 inclusive.
6. A man draws 1 card from each of 2 separate decks. Find the probability (a) that the card from the first deck is a jack and that from the second deck a spade, (b) that the card from the first deck is black and that from the second deck a red ace, (c) that the card from the first deck is a heart and that from the second deck a face card (king, queen, or jack).
7. A man draws 1 card from each of 3 separate decks. What is the probability that all are (a) red? (b) aces, deuces, or either? (c) face cards?
8. What is the probability of throwing 3 aces in a single throw of 3 dice (or, what is equivalent, in 3 throws with 1 die)?
9. A man is 50 years old, his wife 45. Assuming that the probabilities of their deaths are independent, find the probability (a) that both will die within a year, (b) that she will die within a year but that he will not.

10. A man draws 2 cards from a deck. What is the probability

(a) that the first card is a spade and the second a diamond,
(b) that either card is a spade and the other a diamond,
(c) that both cards are spades?

11. A man draws a card from a deck, replaces it, and makes another draw. What is the probability (a) that he gets a club both times? (b) that he gets an ace the first time and a face card the second? (c) that he gets a king both times?

12. What is the probability of a run of 10 consecutive heads in tossing a coin?

13. One box contains 6 white and 6 black balls, a second box contains 10 white and 5 black balls. A card is dealt from a well-shuffled deck and if it is a diamond a ball is drawn from the first box, otherwise a ball is drawn from the second box. What is the probability of obtaining a white ball?

14. One box contains 5 white and 5 black balls, a second box contains 10 white and 2 black balls. A die is thrown and if an ace occurs 3 balls are drawn from the first box, otherwise 3 balls are drawn from the second box. Find the probability of getting 2 white balls and 1 black ball.

152. Probability in repeated trials.

If p is the probability that an event will happen and q is the probability that it will fail in any trial, the probability that it will happen exactly r times in n trials is

$$_nC_r p^r q^{n-r} = \frac{n!}{r!(n-r)!} p^r q^{n-r}. \tag{1}$$

As an illustration, consider the probability of throwing exactly 2 aces in 5 throws of a die. The probability that the event (throwing of an ace) will happen in any trial is $p = \frac{1}{6}$ and the probability that it will fail is $q = \frac{5}{6}$. The probability that it will happen twice and fail three times in a specified order, such as happening the first two times and failing the last three, is by the theorem on independent probabilities,

$$\frac{1}{6}\cdot\frac{1}{6}\cdot\frac{5}{6}\cdot\frac{5}{6}\cdot\frac{5}{6}=\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^3.$$

But the 2 aces may be obtained on any 2 of the 5 throws, that is, in ${}_5C_2 = \dfrac{5\cdot 4}{1\cdot 2} = 10$ ways. If the happening of the event is indicated by h and the failure by f, these 10 ways may be indicated by

hhfff	*hfhff*	*hffhf*	*hfffh*	*fhhff*
fhfhf	*fhffh*	*ffhhf*	*ffhfh*	*fffhh*

The probability for any one of these orders is $(\frac{1}{6})^2(\frac{5}{6})^3$, and the total probability is

$$10\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^3=\frac{625}{3888}.$$

More generally, the probability that an event will happen in each of r specified trials and fail in the remaining $n - r$ is p^rq^{n-r}. But these r trials can be chosen from the n trials in ${}_nC_r$ ways. Consequently, since these ways form a mutually exclusive set, the total probability is ${}_nC_rp^rq^{n-r}$.

It should be noted that this is term number $r + 1$ in the binomial formula

$$(q+p)^n = q^n + {}_nC_1q^{n-1}p + {}_nC_2q^{n-2}p^2 + \cdots \\ + {}_nC_rq^{n-r}p^r + \cdots + p^n. \qquad (2)$$

In fact the terms of this expansion give the probabilities that the event will happen exactly 0, 1, 2, $\cdots$, r, $\cdots$, n times in n trials.

The probability that an event will happen *at least* or *at* *most* a certain number of times in a given number of trials is found by taking the sum of the appropriate terms in (2).

Example 1.

Find the probability of throwing at least 3 aces in 5 throws of a die.

SOLUTION. This is the probability of throwing 3, 4, or 5 aces and is thus

$$ {}_5C_3p^3q^2 + {}_5C_4p^4q + p^5 = 10\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^2 + 5\left(\frac{1}{6}\right)^4\left(\frac{5}{6}\right) + \left(\frac{1}{6}\right)^5 $$
$$ = \frac{250}{6^5} + \frac{25}{6^5} + \frac{1}{6^5} = \frac{276}{7776} = \frac{23}{648}. $$

Example 2.

Find the probability of throwing at least 2 aces in 10 throws of a die.

SOLUTION. This probability is the sum of the probabilities of throwing any number of aces from 2 to 10 inclusive. Instead of summing the nine corresponding terms of the binomial expansion we find the probability of throwing no aces or 1 ace and subtract from 1.

The probability of 0 or 1 ace is

$$ \left(\frac{5}{6}\right)^{10} + 10\left(\frac{5}{6}\right)^9 \cdot \frac{1}{6} = \frac{3 \cdot 5^{10}}{2^{10} \cdot 3^{10}} = \frac{9765625}{20155392}. $$

The probability of throwing at least 2 aces is

$$ 1 - \frac{9765625}{20155392} = \frac{10389767}{20155392}. $$

EXERCISES XVII. C

1. Find the probability, in tossing 5 coins, of getting (a) exactly 2 heads, (b) at most 2 heads.
2. A man makes 4 draws from a deck of cards, each time replacing the card drawn and reshuffling the deck before making

another draw. Find the probability that he gets (a) exactly 3 kings, (b) at least 3 kings.

3. What is the probability, in throwing 6 dice, of getting (a) exactly 3 aces? (b) at least 3 aces?

4. A baseball player has a batting average of 0.300. Assuming that this can be used as the probability that he will get a hit, find the probability (a) that he will get exactly 2 hits in 4 times at bat, (b) that he will get at most 1 hit in 5 times at bat.

5. A rifleman is able, on the average, to hit a target at a certain range 8 times out of 10. Find the probability that in a round of 10 shots he will (a) hit the target every time, (b) make 9 or 10 hits.

6. In a certain manufacturing process it is found that, on the average, 1 article out of 100 is defective. What is the probability that a random sample of 5 articles will contain (a) no defective articles? (b) exactly 1 defective article?

7. Find the probability that of a group of 3 sixty-year-old men, (a) exactly 2 will die during the year, (b) a specified 2 will die during the year, but the third will live, (c) at most 2 will die during the year.

8. According to records in the office of a college registrar, 5 per cent of the students in a certain course fail to pass. What is the probability that in a group of 7 students of the course, selected at random, exactly 2 will fail?

9. Hospital records show that 10 per cent of the cases of a certain disease are fatal. Find the probability that out of 6 patients admitted with this disease (a) all will recover, (b) exactly 2 will die, (c) at least 2 will die.

10. Find the probability, in tossing a coin 10 times, of getting (a) 5 heads and 5 tails, (b) a run of 5 heads followed by a run of 5 tails, (c) the sequence shown in the following schematic representation: *hhhtththth*.

MISCELLANEOUS EXERCISES XVII. D

1. From a box containing 8 white balls and 4 black balls, 3 balls are drawn at random. Find the probability that 2 are white and 1 is black, under the assumption (a) that each ball is

returned to the box before the next is drawn, (b) that the 3 balls are drawn simultaneously.

2. There are 5 keys, only 1 of which will unlock a certain door. If the keys are selected at random, one at a time, what is the probability that the lock can be opened by (a) the first key? (b) the second key? (c) the fifth key? (d) If the first and second keys have been tried and have failed to unlock the door, what is the probability that it can be opened by the third key?

3. Two dice are thrown simultaneously. What is the probability that the number of spots uppermost is 6?

4. Three dice are thrown simultaneously. What is the probability that the number of spots uppermost is 6?

5. The instructor in a class writes the name of each of the 25 pupils on a separate card. He then shuffles the cards thoroughly, draws a card, and calls on the pupil whose name appears on the card to recite. If he repeats this procedure each time before calling on a pupil what is the probability that a pupil who has just recited will be the next one called on?

6. A man has a ticket in a lottery in which 1000 tickets are out. The main prize is \$200 and there are also two prizes of \$50 each and five \$10 prizes. What is his expectation?

7. From a group of 6 boys and 4 girls a committee of 3 is chosen by lot. What is the probability that it will consist of more girls than boys?

8. Five identical disks are marked with the numbers 1, 2, 3, 4, 5, respectively, and placed in a box. After they are thoroughly mixed 3 of them are drawn at random. What is the probability that their average value is 3 (a) if they are drawn simultaneously? (b) if each disk is replaced and mixed with the others before the next drawing is made?

9. A student taking a true-false test consisting of 12 questions guesses at the answers. Assuming that he is equally likely to make a correct or an incorrect guess on each question, find the probability (a) that all of his answers will be right, (b) that half of them will be right and half of them wrong, (c) that 75 per cent or more of them will be right.

10. A multiple-choice test consists of 10 questions. After each

question are listed 4 answers, only 1 of which is correct. If a student attempts to guess the correct answer to each question, what is the probability that more than half of his answers will be right?

11. A and B take turns drawing cards from a deck, the card being replaced before the next draw is made. The first one to draw a spade is to receive \$5. If A draws first what is the expectation of each?

12. Each of the integers from 1 to 9 inclusive is written on a separate card. A and B take turns at drawing the cards, one by one, until all have been drawn. The first to draw the number "1" wins the game. If A has the first turn what is the probability of winning for each?

13. Solve the preceding exercise for the case in which the integers from 1 to 10 inclusive are used.

14. In a baseball World Series the probability that the American League team will win an individual game is p, the probability that the National League team will win being, consequently, $1 - p$. The championship is decided by the "best 4 games out of 7"; that is, the pennant goes to the team which first wins 4 games. Find the probability of winning the championship for each team.

15. There are 8 seats in a row. If 2 seats are chosen at random what is the probability that they are adjacent to each other?

16. Two persons are placed at random at a round table about which there are 8 chairs. What is the probability that they will be next to each other?

17. Solve the preceding exercise for the case of 3 persons.

18. Five hundred tickets, marked 1c., 2c., 3c., and so on up to \$5.00, are placed in a box and thoroughly mixed. What is the expectation of a person who is to receive the amount marked on the ticket which he draws?

19. A slot machine, when it stops, shows 3 cards in a row. Each of these 3 is one of 5 (A, K, Q, J, 10) equally likely to appear. The player of the machine is paid 24 nickels if 3 A's appear, 12 nickels if 3 K's or 3 Q's appear, 8 nickels if 2 A's and a K appear, and 4 nickels if an A, a K, and a Q appear. What is his expectation?

20. It is known that in a certain manufacturing process 1 per cent

of the articles are defective. Find the probability that a random sample of 15 articles will contain (a) no defectives, (b) exactly 1 defective, (c) exactly 2 defectives, (d) not more than 2 defectives.

21. In a clairvoyance experiment 5 cards are placed, face downward, on a table. The person on whom the experiment is being tried knows that 3 of the cards are red and 2 black, but has no other information regarding them. If he tries to designate which cards are red and which are black, but has no powers of clairvoyance, what is the probability that the number of cards whose color he names correctly is (a) 5? (b) 4? (c) 3? (d) 2? (e) 1? (f) 0?

22. If guinea pigs are inoculated with a certain disease only 40 per cent of them recover. Ten of these animals are given the disease and afterwards are treated with a certain drug. Eight recover. Does this indicate that the drug has a curative property? In other words, if it has no such property what is the probability that 8 or more animals would recover anyhow? If this probability is small it would mean that 8 or more recoveries are not likely unless the drug has some curative properties.

23. A man stands on a street corner and tosses a coin. If it falls heads he walks 1 block north, if it falls tails he walks 1 block south. Starting at his new position he repeats this procedure. What is the probability that after having gone through the procedure 10 times he is (a) exactly 4 blocks from his original starting point? (b) exactly 3 blocks south of his original starting point?

24. Each of 4 persons writes, at random, the name of one of the others. Find the probability (a) that all of the names will be written, (b) that the name of a specified person will not be written.

25. Each of n persons writes, at random, the name of one of the others. What is the probability that the name of a specified person will not be written?

CHAPTER XVIII

Determinants

153. Determinants of order two.

Let us solve the pair of equations

$$\begin{aligned} a_1x + b_1y &= k_1, \\ a_2x + b_2y &= k_2. \end{aligned} \qquad (1)$$

Multiply the first by b_2, the second by b_1. This gives

$$\begin{aligned} a_1b_2x + b_1b_2y &= k_1b_2, \\ a_2b_1x + b_1b_2y &= k_2b_1. \end{aligned}$$

Subtracting, we get

$$(a_1b_2 - a_2b_1)x = k_1b_2 - k_2b_1,$$

and, if $a_1b_2 - a_2b_1 \neq 0$,

$$x = \frac{k_1b_2 - k_2b_1}{a_1b_2 - a_2b_1}. \qquad (2)$$

Similarly,

$$y = \frac{k_2a_1 - k_1a_2}{a_1b_2 - a_2b_1}. \qquad (3)$$

That these values satisfy equations (1) can be proved by actual substitution.

The denominators in the expressions for x and y are identical. We denote this common denominator by the symbol

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}, \tag{4}$$

which is called a **determinant.** This determinant, which has two rows and two columns, is said to be of **order two.** The a's and b's are **elements** of the determinant. The elements a_1 and b_2 which lie in the line from the upper left-hand corner to the lower right-hand corner of the determinant constitute the **principal diagonal.** It is seen that the value of the determinant is found by taking the product of the elements in the principal diagonal and subtracting from it the product of the elements in the other diagonal. Thus,

$$\begin{vmatrix} 5 & 4 \\ 2 & 3 \end{vmatrix} = 5 \cdot 3 - 2 \cdot 4 = 15 - 8 = 7,$$

$$\begin{vmatrix} 6 & 4 \\ -2 & 3 \end{vmatrix} = 6 \cdot 3 - (-2) \cdot 4 = 18 + 8 = 26.$$

We can now write the solution (2) and (3) of the pair of equations (1), by means of determinants, as follows:

$$x = \frac{\begin{vmatrix} k_1 & b_1 \\ k_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \qquad y = \frac{\begin{vmatrix} a_1 & k_1 \\ a_2 & k_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}.$$

It may be noted that for the determinant in the denominator, the elements in the first column are the coefficients of x in the original equations (1), and that the elements in the second column are the coefficients of y. When we solve for x, the determinant in the numerator is the same as that in the denominator, except that the coefficients of x are replaced by the right-hand members of equations (1); when we solve for y, the determinant in the numerator is the same as that in the denominator, except that the coefficients of y are replaced by the right-hand members of (1).

Example.

Solve the following pair of equations by determinants:

$$2x - 3y = 16,$$
$$5x + 2y = 2.$$

SOLUTION.

$$x = \frac{\begin{vmatrix} 16 & -3 \\ 2 & 2 \end{vmatrix}}{\begin{vmatrix} 2 & -3 \\ 5 & 2 \end{vmatrix}} = \frac{16 \cdot 2 - 2 \cdot (-3)}{2 \cdot 2 - 5 \cdot (-3)} = \frac{38}{19} = 2,$$

$$y = \frac{\begin{vmatrix} 2 & 16 \\ 5 & 2 \end{vmatrix}}{\begin{vmatrix} 2 & -3 \\ 5 & 2 \end{vmatrix}} = \frac{2 \cdot 2 - 5 \cdot 16}{2 \cdot 2 - 5 \cdot (-3)} = \frac{-76}{19} = -4.$$

EXERCISES XVIII. A

Solve exercises 1–24 of II. B by means of determinants.

154. Determinants of order three.

Systems of linear equations in more than two unknown quantities can also be solved by means of determinants. To illustrate, let us solve the system of equations

$$a_1x + b_1y + c_1z = k_1, \tag{1}$$
$$a_2x + b_2y + c_2z = k_2, \tag{2}$$
$$a_3x + b_3y + c_3z = k_3. \tag{3}$$

The method of obtaining a given equation is shown at the left; thus, our first step is to multiply equation (1) by c_2, which is indicated by $c_2 \cdot (1)$.

$c_2 \cdot (1)$ $$a_1c_2x + b_1c_2y + c_1c_2z = k_1c_2. \tag{4}$$
$c_1 \cdot (2)$ $$a_2c_1x + b_2c_1y + c_1c_2z = k_2c_1. \tag{5}$$
(4) − (5) $$(a_1c_2 - a_2c_1)x + (b_1c_2 - b_2c_1)y = k_1c_2 - k_2c_1. \tag{6}$$

$$c_3 \cdot (2) \qquad a_2c_3x + b_2c_3y + c_2c_3z = k_2c_3. \qquad (7)$$

$$c_2 \cdot (3) \qquad a_3c_2x + b_3c_2y + c_2c_3z = k_3c_2. \qquad (8)$$

$$(7) - (8) \quad (a_2c_3 - a_3c_2)x + (b_2c_3 - b_3c_2)y = k_2c_3 - k_3c_2. \qquad (9)$$

Next eliminate y from (6) and (9), obtaining

$$[(a_1c_2 - a_2c_1)(b_2c_3 - b_3c_2) - (a_2c_3 - a_3c_2)(b_1c_2 - b_2c_1)]x$$
$$= (k_1c_2 - k_2c_1)(b_2c_3 - b_3c_2) - (k_2c_3 - k_3c_2)(b_1c_2 - b_2c_1).$$

Solve for x, and simplify:

$$x = \frac{k_1b_2c_3 - k_1b_3c_2 + k_2b_3c_1 - k_2b_1c_3 + k_3b_1c_2 - k_3b_2c_1}{a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1}. \qquad (10)$$

(It is assumed that the denominator is not zero.)

The denominator of the fraction in (10) may be put in the form

$$a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$$
$$= a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}. \qquad (11)$$

We now introduce the new symbol,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \qquad (12)$$

whose value is defined to be (11). This symbol is a **determinant of order three** (it has three rows and three columns).

The numerator of (10) is similarly transformed into

$$k_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - k_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + k_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$
$$= \begin{vmatrix} k_1 & b_1 & c_1 \\ k_2 & b_2 & c_2 \\ k_3 & b_3 & c_3 \end{vmatrix}, \qquad (13)$$

and we can write the value of x as the quotient of (13) divided by (12). The unknowns y and z can be found as similar quotients, and the complete solution of the linear system (1) can be written as

$$x = \frac{\begin{vmatrix} k_1 & b_1 & c_1 \\ k_2 & b_2 & c_2 \\ k_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}, \qquad y = \frac{\begin{vmatrix} a_1 & k_1 & c_1 \\ a_2 & k_2 & c_2 \\ a_3 & k_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}},$$

$$z = \frac{\begin{vmatrix} a_1 & b_1 & k_1 \\ a_2 & b_2 & k_2 \\ a_3 & b_3 & k_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}. \qquad (14)$$

That (14) is a solution of the given system of equations can be proved by direct substitution.

The common denominator is the determinant of the coefficients of the system of equations (1), (2), (3); that is, its elements are the coefficients of x, y, z in these equations arranged in their respective positions. The numerator in the expression for any unknown is the same as the denominator, except that the coefficients of this unknown are replaced by the corresponding constant right-hand members of the system of equations.

The determinant method of solving a system of linear equations eliminates all but one of the unknown quantities at once. Thus, the foregoing solution of the set of equations (1), (2), (3), for the unknown x, is equivalent to multiplying them by the determinants

$$\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}, \qquad -\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix}, \qquad \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix},$$

respectively, and adding the results. This eliminates y and z simultaneously.

Example.

Solve by determinants:

$$\begin{aligned} 4x - y + 4z &= 2, \\ 2x + 3y + 5z &= 2, \\ 7x - 2y + 6z &= 5. \end{aligned}$$

SOLUTION.

$$x = \frac{\begin{vmatrix} 2 & -1 & 4 \\ 2 & 3 & 5 \\ 5 & -2 & 6 \end{vmatrix}}{\begin{vmatrix} 4 & -1 & 4 \\ 2 & 3 & 5 \\ 7 & -2 & 6 \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} 4 & 2 & 4 \\ 2 & 2 & 5 \\ 7 & 5 & 6 \end{vmatrix}}{\begin{vmatrix} 4 & -1 & 4 \\ 2 & 3 & 5 \\ 7 & -2 & 6 \end{vmatrix}}, \quad z = \frac{\begin{vmatrix} 4 & -1 & 2 \\ 2 & 3 & 2 \\ 7 & -2 & 5 \end{vmatrix}}{\begin{vmatrix} 4 & -1 & 4 \\ 2 & 3 & 5 \\ 7 & -2 & 6 \end{vmatrix}}.$$

The denominator has the value

$$4\begin{vmatrix} 3 & 5 \\ -2 & 6 \end{vmatrix} - 2\begin{vmatrix} -1 & 4 \\ -2 & 6 \end{vmatrix} + 7\begin{vmatrix} -1 & 4 \\ 3 & 5 \end{vmatrix}$$

$$= 4(18 + 10) - 2(-6 + 8) + 7(-5 - 12)$$

$$= 4 \cdot 28 - 2 \cdot 2 + 7(-17) = -11.$$

The numerator of x has the value

$$2\begin{vmatrix} 3 & 5 \\ -2 & 6 \end{vmatrix} - 2\begin{vmatrix} -1 & 4 \\ -2 & 6 \end{vmatrix} + 5\begin{vmatrix} -1 & 4 \\ 3 & 5 \end{vmatrix}$$

$$= 2 \cdot 28 - 2 \cdot 2 + 5(-17) = -33.$$

Thus, $$x = \frac{-33}{-11} = 3.$$

The student should complete the solution for y and z. The results are $y = 2$, $z = -2$.

EXERCISES XVIII. B

Evaluate the following determinants:

1. $\begin{vmatrix} 1 & 4 & 2 \\ 3 & 5 & 1 \\ 2 & 3 & 2 \end{vmatrix}$.

2. $\begin{vmatrix} 2 & -1 & 3 \\ 4 & 7 & -5 \\ 6 & 1 & 2 \end{vmatrix}$.

3. $\begin{vmatrix} 4 & 0 & 3 \\ -5 & 3 & 1 \\ 2 & -6 & 7 \end{vmatrix}$.

4. $\begin{vmatrix} 7 & 11 & -4 \\ 10 & 8 & -3 \\ -5 & 4 & 12 \end{vmatrix}$.

5. $\begin{vmatrix} 27 & 9 & -19 \\ 11 & -15 & 0 \\ 43 & 0 & 22 \end{vmatrix}$.

6. $\begin{vmatrix} 15 & -6 & 20 \\ 13 & -23 & 17 \\ 21 & 14 & -9 \end{vmatrix}$.

7. $\begin{vmatrix} 5 & 8 & 0 \\ -21 & 0 & -2 \\ 0 & 7 & -37 \end{vmatrix}$.

8. $\begin{vmatrix} 19 & 9 & -4 \\ 5 & 1 & -2 \\ 2 & 3 & 1 \end{vmatrix}$.

9. $\begin{vmatrix} 3 & 2 & 1 \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{6} & \frac{1}{12} \end{vmatrix}$.

10. Factor the following determinant:

$$\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}.$$

11. Solve:

$$\begin{vmatrix} 2x & 9 & 6 \\ 4 & 7 & -2 \\ 1 & x & -1 \end{vmatrix} = 0.$$

12. For what values of x will the following determinant be positive?

$$\begin{vmatrix} x & 3 & 1 \\ 1 & 3 & x \\ 0 & x & 2 \end{vmatrix}.$$

Solve for x, y, z by determinants:

13. $5x + 2y + 4z = 4,$
$3x - y + 2z = -11,$
$7x - 3y - 3z = 8.$

14. $3x - 9y - 2z = 25,$
$4x + 8y + 3z = -12,$
$5x + 10y - 2z = 31.$

15. $5x + 7y - 3z = 36,$
$2x - 8y + 4z = -62,$
$3x - y + 9z = 18.$

16. $9x - 6y - 4z = -5,$
$2x - 3y + 3z = 0,$
$11x + y - 10z = 17.$

17. $9x + 4y - 8z = 10,$
$7x - 6y + 3z = 8,$
$3x - 5y - 2z = -8.$

18. $5x + 2y - 3z = 62,$
$6x + 11z = -8,$
$4x + 3y + 8z = 22.$

19. $6x - y + 7z = 3,$
$5x + 8y - 9z = 0,$
$2x - 10y + 5z = 1.$

20. $2x + 3y + 4z = 7,$
$8x + 6y + z = -11,$
$x + 9y + 2z = 11.$

21. $\frac{x}{3} + \frac{y}{4} + \frac{z}{10} = 6,$
$\frac{x}{4} + \frac{y}{2} - \frac{z}{5} = 3,$
$\frac{x}{6} + y + \frac{z}{2} = 11.$

22. $x + y = a,$
$y + z = b,$
$z + x = c.$

23. Solve exercises II. B, 26–35 by means of determinants.

155. Determinants of any order.

The n by n array

$$D = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}, \quad (1)$$

in which there are n^2 elements arranged in n rows and n columns, is a **determinant of order n.** The first subscript of an element is the number of the row in which the element lies, the second subscript is the number of the column. (The element a_{12}, — read "a sub one-two" or "a one-two," not "a twelve" — for example, lies in the first row and the second column.) The value of the determinant may be defined as follows:

(i) *Write down all possible products of n factors each that can be obtained by selecting one and only one element from each row and each column.* (According to section 142 there will be n ! such products.)

(ii) *In each product count the number of pairs of elements* *in which one element appears to the right of and above the other* (when the two elements are in their respective positions in the determinant). Pairs of this type may be called **negative pairs,** pairs of the other type **positive pairs.** *If there is an even number of negative pairs, prefix a plus sign to the product; if there is an odd number, prefix a minus sign.* Since there are no negative pairs in the principal diagonal (the diagonal running from upper left-hand to lower right-hand corner), and zero is an even integer, the plus sign is always to be prefixed to this special product.

The value of the determinant is the algebraic sum of these products. This sum is usually spoken of as the **expansion** of the determinant.

To illustrate the rule of signs, (ii) above, let us consider the product $a_2b_1c_3$ from the third-order determinant (12) of section 154. There are three pairs of elements in this product, namely, a_2b_1, a_2c_3, b_1c_3. Of these, only the first is a negative pair; consequently the product must be prefixed with a minus sign.

EXERCISE

Show that the value of the determinant (12) of section 154, as given by (11) of that section, is precisely the same as the value obtained from the definition of the present section.

156. Properties of determinants.

I. The value of a determinant is unchanged if corresponding rows and columns are interchanged.

Let

$$D = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}, \quad D' = \begin{vmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \cdot & \cdot & \cdot & \cdot \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{vmatrix}.$$

The products obtained from D' are exactly the same as those obtained from D, so that the terms of D' are the same as those of D. However, it is readily seen that any pair of elements which is a positive pair in D is also a positive pair in D'. Consequently, the signs of the corresponding terms in the expansions of the two determinants are alike, and the values of the two determinants are thus identical.

It follows that *for every theorem about the columns of a determinant there exists a corresponding theorem about the rows, and vice versa.*

II. If two columns (or rows) are interchanged the sign of the determinant is changed.

Let us consider first the effect of interchanging two adjacent rows. Obviously the products composing the terms of the determinant will be unchanged, since each is composed of one and only one element from each row and each column. The only pair of elements of any product whose status is changed as far as the rule of signs is concerned is the pair in the rows interchanged. If this pair was a positive pair before the interchange, it will be a negative pair afterwards; if it was a negative pair before, it will be a positive pair afterwards. Hence the interchange transforms the number of negative pairs from even to odd or from odd to even. Thus, the sign of every term in the expansion of the determinant is changed; that is, the sign of the determinant is changed.

Now let us consider the effect of interchanging any two rows. Suppose these two rows have k other rows between them. Their interchange may be effected by $2k + 1$ interchanges of adjacent rows: $k + 1$ to bring the lower row into the position previously occupied by the upper, k more to bring the upper row into the position previously occupied by the lower. But $2k + 1$ is an odd number, and an odd number of interchanges of adjacent rows changes the sign of the determinant.

III. If two columns (or rows) are identical, the value of the determinant is zero.

Let D be the value of the determinant. An interchange of columns will give a determinant whose value is $-D$. But if the identical columns are interchanged, the value of the determinant will obviously be unaffected. Hence $D = -D$, or $2D = 0$, and $D = 0$.

IV. If all the elements of a column (or row) are multiplied by the same number m, the determinant is multiplied by m. That is,

$$\begin{vmatrix} ma_1 & b_1 & \cdots \\ ma_2 & b_2 & \cdots \\ \cdot & \cdot & \cdot \end{vmatrix} = m \begin{vmatrix} a_1 & b_1 & \cdots \\ a_2 & b_2 & \cdots \\ \cdot & \cdot & \cdot \end{vmatrix}.$$

From the definition of the value of a determinant, as given in section 155, it is seen that each term in the expansion of the new determinant is m times the corresponding term in the expansion of the original determinant. That is, the new determinant is m times the original.

V. If every element of a column (or row) is zero, the value of the determinant is zero.

For each term in the expansion of the determinant contains the factor zero.

VI. If each element of any column (or row) is expressed as the sum of two or more terms, the determinant may be expressed as the sum of two or more determinants. Thus,

$$\begin{vmatrix} a_1 + a_1' & b_1 & \cdots \\ a_2 + a_2' & b_2 & \cdots \\ \cdot & \cdot & \cdot \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & \cdots \\ a_2 & b_2 & \cdots \\ \cdot & \cdot & \cdot \end{vmatrix} + \begin{vmatrix} a_1' & b_1 & \cdots \\ a_2' & b_2 & \cdots \\ \cdot & \cdot & \cdot \end{vmatrix}.$$

For any term in the expansion of the determinant on the left is equal to the sum of the corresponding terms of the determinants on the right.

VII. The value of a determinant is unchanged if to the elements of any column (or row) are added the corresponding elements of any other column (or row) each multiplied by the same number m. (Note that m may be negative, also that it may have either of the particular values ± 1.)

For example, consider

$$D = \begin{vmatrix} a_1 & b_1 & \cdots \\ a_2 & b_2 & \cdots \\ \cdot & \cdot & \cdot \end{vmatrix}, \qquad D' = \begin{vmatrix} a_1 + mb_1 & b_1 & \cdots \\ a_2 + mb_2 & b_2 & \cdots \\ \cdot & \cdot & \cdot \end{vmatrix}.$$

By VI,

$$D = \begin{vmatrix} a_1 & b_1 & \cdots \\ a_2 & b_2 & \cdots \\ \cdot & \cdot & \cdot \end{vmatrix} + \begin{vmatrix} mb_1 & b_1 & \cdots \\ mb_2 & b_2 & \cdots \\ \cdot & \cdot & \cdot \end{vmatrix}.$$

The first determinant on the right-hand side of this last equation is D; if by IV we factor m from the second, it has two columns identical, and hence by III is zero. Therefore, $D = D'$.

157. Expansion of a determinant by minors.

If in any determinant the row and the column containing a given element are removed, the determinant formed by the remaining elements is called the **minor** of the given element. For example, in the determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},$$

the minor of b_1 is $$\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}.$$

We shall now show how a determinant of any order can be expanded by means of minors.

THEOREM I. *If the element in the upper left-hand corner of a determinant is a_{11}, the sum of the terms involving a_{11} in the expansion of the determinant is $a_{11}A_{11}$, where A_{11} is the minor of a_{11}.*

For each term of the determinant involving a_{11} is obtained by multiplying a_{11} by one and only one element from each of the remaining rows and columns, that is, by a term of its minor, A_{11}. Furthermore, the sign of each term of the original determinant involving a_{11} is the same as that of the corresponding term obtained by multiplying a_{11} by the appropriate term of A_{11}, since a_{11} forms a positive pair with each and every element of A_{11}.

THEOREM II. *If the element in the ith row and jth column of a determinant is a_{ij}, the sum of the terms involving a_{ij} in the expansion is $(-1)^{i+j}a_{ij}A_{ij}$, where A_{ij} is the minor of a_{ij}.*

The element a_{ij} can be carried into the upper left-hand corner of the determinant by interchanging the ith row with each preceding row in turn, and then interchanging the jth column with each preceding column in turn. This will not disturb the elements in the minor A_{ij}. The sign of the determinant will have been changed $(i - 1) + (j - 1)$ times, since the process consists of this number of interchanges of adjacent rows or columns. Thus, if D is the value of the original determinant and D' is the value of the determinant which has a_{ij} in the upper left-hand corner, then

$$D' = (-1)^{i-1+j-1}D = (-1)^{i+j-2}D = (-1)^{i+j}D.$$

But, by Theorem I, the sum of all terms in the expansion of D' involving a_{ij} is equal to a_{ij} times its minor in D', which is A_{ij}. Hence the sum of terms in D involving a_{ij} is $(-1)^{i+j}a_{ij}A_{ij}$.

Combining Theorems I and II, we have the following method of expanding a determinant according to the elements of a column (or row):

Multiply each element in the given column (or row) by its minor, and prefix a plus or minus sign according as the sum of the number of the row and the number of the column in which the element lies is even or odd. The value of the determinant is the algebraic sum of these products.

The value of a determinant can be expressed, thus, in a variety of ways. For example, if capital letters represent the minors of the elements denoted by the corresponding small letters, we have

$$\begin{vmatrix} a_1 & b_1 & c_1 & \cdots \\ a_2 & b_2 & c_2 & \cdots \\ a_3 & b_3 & c_3 & \cdots \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix} = \begin{cases} a_1A_1 - a_2A_2 + a_3A_3 - \cdots, & (1) \\ \text{or } -b_1B_1 + b_2B_2 - b_3B_3 + \cdots, & (2) \\ \cdots\cdots\cdots\cdots & \\ \text{or } a_1A_1 - b_1B_1 + c_1C_1 - \cdots, & (3) \\ \text{or } -a_2A_2 + b_2B_2 - c_2C_2 + \cdots, & (4) \\ \cdots\cdots\cdots\cdots & \end{cases}$$

The expression (1) is called the **expansion, or development, according to the first column,** (2) is the development according to the second column, (3) and (4) are developments according to the first and second rows respectively.

Example 1.

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 & d_1 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix}$$

$$+ a_3 \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_4 & c_4 & d_4 \end{vmatrix} - a_4 \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix}.$$

The development can be completed by expanding the third-order determinants.

In a numerical case, we can often simplify the process of evaluation by making use of some of the properties developed earlier in the chapter.

Example 2.

Evaluate the determinant

$$\begin{vmatrix} 4 & -3 & 1 & 0 \\ 2 & 3 & 6 & 7 \\ 6 & -5 & 4 & 5 \\ -2 & 1 & 2 & 4 \end{vmatrix}.$$

Solution. Factor 2 from 1st column (Property IV):

$$\begin{vmatrix} 4 & -3 & 1 & 0 \\ 2 & 3 & 6 & 7 \\ 6 & -5 & 4 & 5 \\ -2 & 1 & 2 & 4 \end{vmatrix} = 2\begin{vmatrix} 2 & -3 & 1 & 0 \\ 1 & 3 & 6 & 7 \\ 3 & -5 & 4 & 5 \\ -1 & 1 & 2 & 4 \end{vmatrix}.$$

Select element 1 in 1st row as a convenient one to use in reducing to zero the other elements in that row. Subtract twice 3rd column from 1st column (Property VII):

$$2\begin{vmatrix} 2-2\cdot 1 & -3 & 1 & 0 \\ 1-2\cdot 6 & 3 & 6 & 7 \\ 3-2\cdot 4 & -5 & 4 & 5 \\ -1-2\cdot 2 & 1 & 2 & 4 \end{vmatrix} = 2\begin{vmatrix} 0 & -3 & 1 & 0 \\ -11 & 3 & 6 & 7 \\ -5 & -5 & 4 & 5 \\ -5 & 1 & 2 & 4 \end{vmatrix}.$$

Add 3 times 3rd column to 2nd column:

$$2\begin{vmatrix} 0 & -3+3\cdot 1 & 1 & 0 \\ -11 & 3+3\cdot 6 & 6 & 7 \\ -5 & -5+3\cdot 4 & 4 & 5 \\ -5 & 1+3\cdot 2 & 2 & 4 \end{vmatrix} = 2\begin{vmatrix} 0 & 0 & 1 & 0 \\ -11 & 21 & 6 & 7 \\ -5 & 7 & 4 & 5 \\ -5 & 7 & 2 & 4 \end{vmatrix}.$$

Expand according to 1st row. Only one term in this expansion remains — as was planned — all other terms have the value zero.

$$2\begin{vmatrix} -11 & 21 & 7 \\ -5 & 7 & 5 \\ -5 & 7 & 4 \end{vmatrix} = -2\begin{vmatrix} 11 & 21 & 7 \\ 5 & 7 & 5 \\ 5 & 7 & 4 \end{vmatrix}.$$

Subtract 2nd row from 3rd:

$$-2\begin{vmatrix} 11 & 21 & 7 \\ 5 & 7 & 5 \\ 0 & 0 & -1 \end{vmatrix} = 2(11\cdot 7 - 21\cdot 5) = 2(-28) = -56.$$

EXERCISES XVIII. C

Evaluate the following determinants by first taking out factors common to rows or columns:

1. $\begin{vmatrix} 20 & 35 & 120 \\ -8 & 21 & 36 \\ 16 & -14 & 48 \end{vmatrix}$.

2. $\begin{vmatrix} xyz & x^2y & xz^2 \\ y & xy & yz \\ yz & xz & z^2 \end{vmatrix}$.

Evaluate the following determinants:

3. $\begin{vmatrix} 1 & -2 & 0 & 3 \\ -1 & 0 & 1 & -1 \\ 2 & 0 & 3 & 2 \\ 1 & 1 & 2 & 0 \end{vmatrix}$.

4. $\begin{vmatrix} 2 & 3 & 0 & 1 \\ -1 & 4 & 3 & 5 \\ 0 & 6 & 4 & -2 \\ 0 & 2 & 1 & 2 \end{vmatrix}$.

5. $\begin{vmatrix} 2 & -1 & 3 & 0 \\ -1 & 3 & -2 & 1 \\ 1 & 0 & -1 & -2 \\ 2 & 3 & 0 & 1 \end{vmatrix}$.

6. $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{vmatrix}$.

7. $\begin{vmatrix} 9 & 5 & 1 & 6 \\ 8 & 4 & 4 & 7 \\ 7 & 3 & 7 & 8 \\ 6 & 2 & 9 & 9 \end{vmatrix}$.

8. $\begin{vmatrix} 4 & -1 & 1 & 5 \\ 2 & -3 & 5 & 1 \\ 1 & 1 & -2 & 2 \\ 5 & 0 & -1 & 2 \end{vmatrix}$.

9. $\begin{vmatrix} 1 & 1 & 3 & 0 \\ -1 & 3 & 2 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & -2 & 2 & -2 \end{vmatrix}$.

10. $\begin{vmatrix} 20 & 70 & 55 & -15 \\ 3 & 8 & 7 & 5 \\ 6 & 16 & 9 & 8 \\ 2 & 10 & 8 & 4 \end{vmatrix}$.

11. $\begin{vmatrix} 2 & 7 & 1 & -1 \\ 4 & -2 & 0 & 8 \\ 3 & -1 & 4 & 6 \\ 4 & 4 & 1 & 2 \end{vmatrix}$.

12. $\begin{vmatrix} -9 & 3 & 8 & 4 \\ 3 & 6 & -2 & 5 \\ 1 & -7 & 6 & 8 \\ 11 & 9 & 14 & -3 \end{vmatrix}$.

13. $\begin{vmatrix} 13 & 27 & 44 & 15 \\ 4 & 8 & -10 & -2 \\ 9 & 18 & 30 & 12 \\ 17 & 35 & 21 & 0 \end{vmatrix}$.

14. $\begin{vmatrix} a & 0 & 0 & 0 \\ b & c & 0 & 0 \\ d & e & f & 0 \\ g & h & i & j \end{vmatrix}$.

15. $\begin{vmatrix} 2 & 0 & 1 & -3 & 0 \\ 5 & 0 & 2 & 1 & -1 \\ 1 & 2 & -3 & 4 & 1 \\ 2 & 4 & 1 & 1 & 1 \\ -1 & 0 & 2 & 4 & 3 \end{vmatrix}$.

16. $\begin{vmatrix} 36 & 24 & -48 & 60 & 120 \\ 27 & 10 & 8 & 7 & 25 \\ -9 & 8 & 20 & 6 & 45 \\ 45 & 14 & 16 & -1 & 35 \\ 12 & 4 & -4 & -3 & 15 \end{vmatrix}.$

158. Application to the solution of linear equations.

We have already seen that systems of linear equations in two and three unknowns can be solved by means of determinants. (See sections 153 and 154.) In order to show that the method is perfectly general, we shall find it necessary to use the following theorem:

THEOREM III. *If in the expansion of a determinant according to a given column (or row), the elements of the given column (or row) are replaced by the elements of another column (or row), the resulting expression is identically equal to zero.*

For example,

$$\begin{vmatrix} a_1 & b_1 & c_1 & \cdots \\ a_2 & b_2 & c_2 & \cdots \\ a_3 & b_3 & c_3 & \cdots \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix} = a_1A_1 - a_2A_2 + a_3A_3 - \cdots,$$

in which the expansion is according to the first column. If we replace the a's of the right-hand member by the corresponding b's, we have

$$b_1A_1 - b_2A_2 + b_3A_3 - \cdots,$$

which is the expansion of

$$\begin{vmatrix} b_1 & b_1 & c_1 & \cdots \\ b_2 & b_2 & c_2 & \cdots \\ b_3 & b_3 & c_3 & \cdots \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix}.$$

This last determinant, however, has two columns alike, and is consequently identically equal to zero.

Let us now consider the following system of n linear equations in n unknowns:

$$\begin{aligned} a_1x + b_1y + c_1z + \cdots &= k_1, \\ a_2x + b_2y + c_2z + \cdots &= k_2, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad &\cdot \quad \cdot \quad \cdot \\ a_nx + b_ny + c_nz + \cdots &= k_n. \end{aligned} \tag{1}$$

To obtain the solution, if there is one, we multiply the first equation by A_1, the second by $-A_2$, and so on, where the A's are the minors of the a's. Adding, we get

$$\begin{aligned} &(a_1A_1 - a_2A_2 + a_3A_3 - \cdots)x + (b_1A_1 - b_2A_2 + b_3A_3 \\ &\quad - \cdots)y + (c_1A_1 - c_2A_2 + c_3A_3 - \cdots)z + \cdots \\ &\quad = k_1A_1 - k_2A_2 + \cdots. \end{aligned} \tag{2}$$

The coefficient of x is the expansion, according to its first column, of the determinant of the coefficients of the system of equations, namely,

$$D = \begin{vmatrix} a_1 & b_1 & c_1 & \cdots \\ a_2 & b_2 & c_2 & \cdots \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix}.$$

The coefficients of the other unknowns are zero by Theorem III. The expression on the right of (2) is the expansion of the determinant

$$D_1 = \begin{vmatrix} k_1 & b_1 & c_1 & \cdots \\ k_2 & b_2 & c_2 & \cdots \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix},$$

which is obtained from D by replacing the coefficients of x (that is, the first column) by the right-hand members of equations (1).

Under the assumption that $D \neq 0$ we find

$$x = \frac{D_1}{D}.$$

Similarly,

$$y = \frac{D_2}{D}, \qquad z = \frac{D_3}{D}, \quad \cdots,$$

where D_2 is the determinant obtained from D by replacing the coefficients of y (second column) by the k's of (1), D_3 is the determinant obtained from D by replacing the coefficients of z by the k's, and so on.

Thus, if there is a solution of the set of equations (1), there is only one, and it is $x = D_1/D$, $y = D_2/D$, $z = D_3/D$, $\cdots$. It can be proved by actual substitution that this really is a solution.

159. Inconsistent and dependent equations.

The solution given in the preceding section breaks down if $D = 0$, since division by zero is meaningless. *If $D = 0$ and any $D_i \neq 0$, the equations have no solution.*

Suppose, for example, that $D = 0$, $D_1 \neq 0$, and that we assume the existence of a solution $x = x_0$, $y = y_0$, $z = z_0$, $\cdots$. We should then have $x D = D_1$. But if $D = 0$ and $D_1 \neq 0$ we have a contradiction, so that the assumption of the existence of a solution is false.

A system of equations having no solution is said to be **inconsistent.** It is obvious from the preceding paragraph that equations (1) of section 158 are inconsistent if $D = 0$ and any $D_i \neq 0$. If $D = 0$ and every $D_i = 0$, the equations may or may not have a solution; that is, they may be consistent or they may be inconsistent.

When a system of equations has an indefinite number of solutions the system is said to be **dependent.**

Example 1.

Show that the following equations are inconsistent:

$$\begin{aligned} 2x - 3y + 5z &= 3, \\ 4x - y + z &= 1, \\ 3x - 2y + 3z &= 4. \end{aligned}$$

SOLUTION.

$$D = \begin{vmatrix} 2 & -3 & 5 \\ 4 & -1 & 1 \\ 3 & -2 & 3 \end{vmatrix} = 0, \qquad D_1 = \begin{vmatrix} 3 & -3 & 5 \\ 1 & -1 & 1 \\ 4 & -2 & 3 \end{vmatrix} = 4.$$

It is not necessary to find D_2 and D_3, since if D is zero and even one of the D_i is different from zero the equations are inconsistent.

Example 2.

Solve the following equations and show that they are dependent:

$$3x - y + 2z = 1, \qquad (1)$$
$$x + y + z = 3, \qquad (2)$$
$$5x + y + 4z = 7. \qquad (3)$$

SOLUTION. Add (1) and (2):

$$4x + 3z = 4, \quad x = \frac{4 - 3z}{4}. \qquad (4)$$

Substitute in (2):

$$\frac{4 - 3z}{4} + y + z = 3, \quad y = \frac{8 - z}{4}. \qquad (5)$$

These values of x and y in terms of z will be found to satisfy the system of equations (1), (2), (3). Since any number of values may be given to z, we can obtain an indefinite number of solutions, and the equations are dependent. (The third may in fact be obtained by multiplying the second by 2 and adding it to the first.) Observe that the solution may be written

$$x = \frac{4 - 3c}{4}; \quad y = \frac{8 - c}{4}, \quad z = c, \quad c \text{ arbitrary}.$$

Note that $D = D_1 = D_2 = D_3 = 0$.

Example 3.

The following equations are inconsistent, even though $D = D_1 = D_2 = D_3 = 0$:

$$\begin{aligned} 3x - y + 2z &= 1, \\ 6x - 2y + 4z &= 3, \\ 9x - 3y + 6z &= 0. \end{aligned}$$

160. Linear systems with more equations than unknowns.

If a linear system has more equations than unknowns, a solution does not in general exist; that is, the system is usually inconsistent.

However, if a solution of certain of the equations can be found, and if this solution satisfies all of the remaining equations, the system is consistent.

If the number of equations is one more than the number of unknowns, a necessary condition for the consistency of the system can easily be derived. For simplicity let us consider the case of three equations in two unknowns:

$$a_1x + b_1y + c_1 = 0, \tag{1}$$
$$a_2x + b_2y + c_2 = 0, \tag{2}$$
$$a_3x + b_3y + c_3 = 0. \tag{3}$$

(Note that the constant terms have been written on the left-hand sides of the equations.)

Suppose that equations (2) and (3) are consistent. Their solution will be

$$x = \frac{\begin{vmatrix} -c_2 & b_2 \\ -c_3 & b_3 \end{vmatrix}}{\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}} = \frac{\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}} = \frac{A_1}{C_1}, \tag{4}$$

$$y = \frac{\begin{vmatrix} a_2 & -c_2 \\ a_3 & -c_3 \end{vmatrix}}{\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}} = -\frac{\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}} = \frac{-B_1}{C_1}. \tag{5}$$

Here A_1, B_1, C_1 are minors of a_1, b_1, c_1 respectively, in the determinant composed of the coefficients and the constant terms of equations (1), (2), (3), namely,

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}. \tag{6}$$

If the system of equations (1), (2), (3) is consistent, the values of x and y, as given by (4) and (5) respectively, which are solutions of (2) and (3), must also satisfy (1). Substituting them in (1), we must have

$$a_1 \frac{A_1}{C_1} - b_1 \frac{B_1}{C_1} + c_1 = 0, \tag{7}$$

or, clearing of fractions,

$$a_1A_1 - b_1B_1 + c_1C_1 = 0 \tag{8}$$

But the left side of (8) is the development of the determinant D according to its first row. Thus, *if the system* (1), (2), (3) *of three linear equations in two unknowns is consistent, we must have* $D = 0$.

The corresponding theorem for the general case can be proved by the same method.

161. Homogeneous equations.

A **homogeneous** equation is one in which all terms are of the same degree, otherwise it is **non-homogeneous. Homogeneous linear** equations are equations all of whose terms are of the first degree, for example.

$$x + 2y = 0, \qquad 3x - 2y + 7z = 0.$$

A system of homogeneous linear equations

$$\begin{aligned} a_1x + b_1y + c_1z + \cdots &= 0, \\ a_2x + b_2y + c_2z + \cdots &= 0, \\ \cdots\cdots\cdots\cdots \\ a_nx + b_ny + c_nz + \cdots &= 0, \end{aligned}$$

always has the **trivial solution** $x = 0, y = 0, \cdots$, as is obvious if these values are substituted in the equations.

If the number of equations is equal to the number of unknowns, and if D, the determinant of the coefficients, is not equal to zero, this is the only solution; if $D = 0$ the equations are dependent and the system **has non-trivial solutions** as well.

Example.

Solve the system of homogeneous equations:

$$x + 2y + 3z = 0, \quad (1)$$
$$2x - y + 2z = 0, \quad (2)$$
$$5x + 7z = 0. \quad (3)$$

SOLUTION. We find

$$D = \begin{vmatrix} 1 & 2 & 3 \\ 2 & -1 & 2 \\ 5 & 0 & 7 \end{vmatrix} = 0.$$

Thus, there are non-trivial solutions.

Multiply (2) by 2 and add to (1):

$$5x + 7z = 0.$$

This is the same as (3), and yields $x = -\frac{7}{5}z$. From (2),

$$y = 2x + 2z = 2\left(-\frac{7}{5}z\right) + 2z = -\frac{4}{5}z.$$

By giving values to z, we can find any number of corresponding values for x and y. For example, if $z = -5$, then $x = 7$, $y = 4$. All solutions are in the continued proportion

$$x : y : z = 7 : 4 : -5.$$

For a complete discussion of systems of linear equations, the student is referred to Maxime Bôcher, *Introduction to Higher Algebra*.

EXERCISES XVIII. D

Solve the following equations by means of determinants:

1. $x + y + z = -2,$
$y + z + t = 3,$
$z + t + x = 0,$
$t + x + y = 5.$

2. $2x + 2y = 3,$
$3y - 2z = 6,$
$5z - 4t = -12,$
$6t + z = 1.$

3. $4x + 3y + z + 2t = 3,$
$6x + y + z + 3t = 30,$
$4x + 2y + z + 2t = 10,$
$3x + 6y + 2z + t = -28.$

4. $5x + 2y + z - 6t = 79,$
$4x - 7y + 2z + 8t = -48,$
$2x + 10y - 4z + t = 48,$
$x + 3y + z + 9t = -35.$

5. $4A - 3B - 2C + 7D = -2,$
$3A - 4B - 7C - 3D = -6,$
$3A + 2B - C - 4D = 13,$
$5A + 5B - 3C + D = 13.$

6. $4x - 6y + 5z - 4w = -4,$
$x - 2y + 2z + 5w = 9,$
$2x + 8y - z + 3w = 13,$
$3x - 14y + 3z + 2w = 14.$

7. $x + y - z = 14,$
$y + z - t = -4,$
$z + t - w = 3,$
$t + w - x = -3,$
$w + x - y = 2.$

8. $x + 2y + 3z + 4t + 5w = 14,$
$2x - 3y + 4z - 5t + 6w = -10,$
$x + y + z + t + w = 4,$
$3x + 4y + 5z + 2t + w = 0,$
$4x + 3y - 2z - t - 6w = 0.$

Test the following equations for consistency, and solve when possible:

9. $x + 2y + 3z = 10,$
$4x + 5y + 6z = 11,$
$7x + 8y + 9z = 12.$

10. $2x + 3y - 9z = 3,$
$x - 5y + 2z = -5,$
$4x - 7y - 5z = -7.$

11. $3x - 2y - 8z = -3,$
$4x + 5y - 3z = 19,$
$5x - 4y - 14z = -7.$

12. $x - 2y + 5z = 0,$
$3x + y + 2z = 15,$
$9x - 4y + 19z = 10.$

13. $5x + 3y - z = 1,$
$3x - 4y - 18z = -3,$
$8x + 7y + 5z = 4.$

14. $x + 2y + 3z = 0,$
$x + 3y + 5z = 1,$
$3x + 5y + 7z = 2.$

15. $x + y + z = 1,$
$3x - 2y + z = 0,$
$10x + 15y + 12z = 13.$

16. $6x + 6y + z = -3,$
$18x - 12y - 17z = -39,$
$42x + 30y - z = -33.$

17. $3x - 2y = -1,$
$4x + 5y = 60,$
$7x - 3y = 11.$

18. $6x - 5y = 44,$
$8x + 11y = -65,$
$10x + 3y = -6.$

19. $3x + 2y = 1,$
$5x - 6y = 39,$
$2x - 7y = 0.$

20. $2x + 9y = 24,$
$11x - 7y = 19,$
$9x - 2y = 23.$

21. $5x - 3y = 31,$
$4x + 7y = -41,$
$11x - 8y = 78.$

22. $4x - 5y = 21,$
$-8x + 3y = -7,$
$6x + 7y = -41.$

23. $2x + 3y = 4,$
$5x + 6y = 7,$
$8x + 9y = 10.$

24. $2x - 3y = 12,$
$5x + 4y = 7,$
$3x + 2y = 3.$

25. $3x + 2y - 8z = 20,$
$5x - 3y + 7z = -6,$
$9x - 11y + 5z = -20,$
$-8x + 2y + 6z = -16.$

26. $2x - 3y = 4,$
$3y - 4z = 5,$
$4z - 5x = 6,$
$5x - 6y + 7z = 8.$

Obtain non-trivial solutions, if they exist, of the following equations:

27. $3x + 4y - 5z = 0,$
$7x - 2y + z = 0,$
$4x + 10y + 9z = 0.$

28. $5x + 3y - z = 0,$
$8x + 7y + 5z = 0,$
$3x - 4y - 18z = 0.$

29. $2x + 6y - z = 0,$
$18x - 24y + 43z = 0,$
$18x + 3y + 25z = 0.$

30. $2x + 3y - 10z = 0,$
$x - 9y - 8z = 0,$
$7x + 11y + 22z = 0.$

31. $25x + 3y - 2z = 0,$
$12x - 4y - 3z = 0,$
$31x + 5y - 2z = 0.$

32. $x + y = 0,$
$y - 2z = 0,$
$3z + 5t = 0,$
$6t - 7x = 0.$

CHAPTER XIX

Partial Fractions

162. Partial fractions.

An algebraic **rational fraction** is the quotient of two polynomials. In elementary algebra the student learns to combine such fractions into a single fraction whose denominator is the lowest common denominator of the separate fractions. Thus,

$$\frac{3}{x-2} + \frac{2}{x+1} = \frac{5x-1}{(x-2)(x+1)}.$$

In this chapter it will be shown how to perform the inverse process of resolving a rational fraction into a sum of fractions, called **partial fractions,** whose denominators are of lower degree. This inverse process is often useful, particularly in calculus.

A rational fraction is **proper** if its numerator is of lower degree than its denominator. An improper fraction can always be reduced to the sum of a polynomial and a proper fraction by dividing the denominator into the numerator until the remainder is of lower degree than the denominator; for example,

$$\frac{2x^3 + x^2 + 2}{x^2 - 1} = 2x + 1 + \frac{2x+3}{x^2-1}.$$

We shall, except for several exercises at the end of the chapter, consider only proper fractions which have been reduced to lowest terms, that is, having no common factor in numerator and denominator.

Methods of resolving rational fractions into partial fractions will be given without proofs. For proofs that the methods are universally valid the reader is referred to more advanced treatises.

Four different cases will be considered, and will be explained by means of examples.

163. Case 1. Factors of the denominator linear, none repeated.

For each factor of the denominator we must have a partial fraction having that factor as denominator and having a constant numerator.

Example.

Resolve into partial fractions:

$$\frac{x+21}{(x-5)(2x+3)}. \tag{1}$$

Solution. Let

$$\frac{x+21}{(x-5)(2x+3)} \equiv \frac{A}{x-5} + \frac{B}{2x+3}. \tag{2}$$

We must now determine the values of the constants A and B so that both members of (2) will be equal for all values of x except those for which some of the denominators are zero. Clear (2) of fractions:

$$x + 21 \equiv A(2x+3) + B(x-5). \tag{3}$$

Method 1. Since both members of (3) are equal for all values of x except possibly 5 and $-\frac{3}{2}$, by the corollary of section 109 they are equal for all values of x, including 5 and $-\frac{3}{2}$. Set $x = 5$ in (3) (to make the coefficient of B equal to zero):

$$26 = 13A, \qquad A = 2.$$

Set $x = -\frac{3}{2}$ in (3) (to make the coefficient of A equal to zero):

$$\frac{39}{2} = -\frac{13}{2}B, \qquad B = -3.$$

Thus,

$$\frac{x + 21}{(x - 5)(2x + 3)} \equiv \frac{2}{x - 5} - \frac{3}{2x + 3}, \tag{4}$$

as can be verified by combining the fractions in the right member of (4).

Two values other than 5 and $-\frac{3}{2}$ could have been given to x, and two equations in A and B obtained and solved. The advantage of assigning these values that cause the factors of the denominator to vanish is that the resulting equations contain only A, and only B, respectively, and can be solved for these constants directly.

Method 2. Expand (3) and collect terms:

$$x + 21 \equiv (2A + B)x + 3A - 5B. \tag{5}$$

Since this is an identity in x, the coefficients of like powers of x are equal by the corollary of section 109. Thus,

$$\begin{aligned} 2A + \ B &= \ 1, \\ 3A - 5B &= 21. \end{aligned} \tag{6}$$

Equations (6) have the solution $A = 2$, $B = -3$, yielding the same result as Method 1.

164. Case 2. Factors of the denominator linear, some repeated.

For each repeated factor such as $(x - a)^k$ we must have a series of partial fractions:

$$\frac{A_1}{x - a} + \frac{A_2}{(x - a)^2} + \cdots + \frac{A_k}{(x - a)^k}.$$

The constant A_k is not zero; the other A's may or may not be.

Example

Resolve into partial fractions:

$$\frac{7x^2 + 3x + 2}{(x-2)(x+1)^2}. \tag{1}$$

SOLUTION. Let

$$\frac{7x^2 + 3x + 2}{(x-2)(x+1)^2} \equiv \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-2}. \tag{2}$$

Clear of fractions:

$$7x^2 + 3x + 2 \equiv A(x+1)(x-2) + B(x-2) + C(x+1)^2.$$

Set $x = -1$: $\quad 6 = -3B, \quad B = -2.$
Set $x = 2$: $\quad 36 = 9C, \quad C = 4.$
Set $x = 0$: $\quad 2 = -2A - 2B + C,$
$\quad 2A = -2 - 2B + C = 6, \quad A = 3.$

Substituting these values of A, B, and C in (2) we get

$$\frac{7x^2 + 3x + 2}{(x-2)(x+1)^2} \equiv \frac{3}{x+1} - \frac{2}{(x+1)^2} + \frac{4}{x-2}.$$

We have used Method 1. Note that after setting $x = -1$ and $x = 2$, which values cause the factors of the denominator to vanish, it is necessary to assign a third value to x since there are three constants (A, B, C) to be obtained and three equations are necessary. The third value chosen was $x = 0$ as this value is extremely easy to substitute.

165. Case 3. Factors of the denominator quadratic, none repeated.

By a quadratic factor we mean here a factor like $x^2 + 4$ or $x^2 - x + 2$, which cannot be factored further into real linear factors. For each such factor, such as $ax^2 + bx + c$, we must have a partial fraction whose denominator is that factor and whose numerator is linear, namely,

$$\frac{Ax + B}{ax^2 + bx + c}.$$

Here A or B (but not both) may be zero.

Example.

Separate into partial fractions:

$$\frac{4x^2 - 5x + 11}{(x - 1)(x^2 + 4)}. \tag{1}$$

SOLUTION. Let

$$\frac{4x^2 - 5x + 11}{(x - 1)(x^2 + 4)} \equiv \frac{A}{x - 1} + \frac{Bx + C}{x^2 + 4}. \tag{2}$$

Clear of fractions:

$$4x^2 - 5x + 11 \equiv A(x^2 + 4) + (Bx + C)(x - 1), \tag{3}$$
$$4x^2 - 5x + 11 \equiv (A + B)x^2 - (B - C)x + 4A - C. \tag{4}$$

We shall use a combination of Methods 1 and 2. Set $x = 1$ in (3):

$$10 = 5A, \qquad A = 2.$$

Equating coefficients of like powers of x on the opposite sides of (4), we get

$$\begin{aligned} A + B &= 4, \\ B - C &= 5, \\ 4A - C &= 11. \end{aligned}$$

Only two of these equations are needed, since we have already found $A = 2$. We readily find $B = 2$, $C = -3$. Thus, (2) becomes

$$\frac{4x^2 - 5x + 11}{(x - 1)(x^2 + 4)} \equiv \frac{2}{x - 1} + \frac{2x - 3}{x^2 + 4}. \tag{5}$$

This exercise can also be solved under Case 1 if we use the imaginary factors of $x^2 + 4$, viz., $(x + 2i)(x - 2i)$. Thus, let

$$\frac{4x^2 - 5x + 11}{(x - 1)(x^2 + 4)} \equiv \frac{D}{x - 1} + \frac{E}{x + 2i} + \frac{F}{x - 2i}. \qquad (6)$$

$$4x^2 - 5x + 11 \equiv D(x + 2i)(x - 2i) + E(x - 1)(x - 2i) + F(x - 1)(x + 2i).$$

Set $x = 1$: $\quad 10 = 5D, \quad D = 2.$

Set $x = 2i$: $\quad -5 - 10i = (-8 - 4i)F,$

$$F = \frac{5(1 + 2i)}{4(2 + i)} = \frac{4 + 3i}{4} = 1 + \frac{3}{4}i.$$

Set $x = -2i$: $\quad -5 + 10i = (-8 + 4i)E,$

$$E = \frac{5(1 - 2i)}{4(2 - i)} = \frac{4 - 3i}{4} = 1 - \frac{3}{4}i.$$

Substituting these values in (6), we get

$$\frac{4x^2 - 5x + 11}{(x - 1)(x^2 + 4)} \equiv \frac{2}{x - 1} + \frac{1 - \frac{3}{4}i}{x + 2i} + \frac{1 + \frac{3}{4}i}{x - 2i}. \qquad (7)$$

We can combine the last two fractions into the single fraction

$$\frac{2x - 3}{x^2 + 4},$$

and (7) reduces to (5).

166. Case 4. Factors of the denominator quadratic, some repeated.

For each repeated quadratic factor such as $(ax^2 + bx + c)^k$ we have a series of partial fractions

$$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_kx + B_k}{(ax^2 + bx + c)^k},$$

in which A_k and B_k are not both zero.

Example.

Resolve into partial fractions:

$$\frac{3x^4 - 4x^3 + 13x^2 - 11x + 23}{(x-3)(x^2+2)^2}. \tag{1}$$

SOLUTION. Let the fraction (1) be identically equal to

$$\frac{A}{x-3} + \frac{Bx+C}{x^2+2} + \frac{Dx+E}{(x^2+2)^2}. \tag{2}$$

Clear of fractions:

$$\begin{aligned} 3x^4 - 4x^3 &+ 13x^2 - 11x + 23 \\ &\equiv A(x^2+2)^2 + (Bx+C)(x-3)(x^2+2) \\ &\quad + (Dx+E)(x-3) \\ &\equiv (A+B)x^4 - (3B-C)x^3 + (4A+2B-3C+D)x^2 \\ &\quad + (-6B+2C-3D+E)x + 4A - 6C - 3E. \end{aligned} \tag{3}$$

Equate coefficients of like powers of x:

$$\begin{aligned} A + B &= 3, \\ 3B - C &= 4, \\ 4A + 2B - 3C + D &= 13, \\ 6B - 2C + 3D - E &= 11, \\ 4A - 6C - 3E &= 23. \end{aligned}$$

These equations have the solution

$$A = 2, \qquad B = 1, \qquad C = -1, \qquad D = 0, \qquad E = -3.$$

Consequently from (1) and (2) we find

$$\frac{3x^4 - 4x^3 + 13x^2 - 11x + 23}{(x-3)(x^2+2)^2} \equiv \frac{2}{x-3} + \frac{x-1}{x^2+2} - \frac{3}{(x^2+2)^2}.$$

In this exercise a combination of Methods 1 and 2 might be considered preferable: Substitute $x = 3$ in (3) to find A, then proceed as above.

EXERCISES XIX. A

Resolve into partial fractions:

1. $\dfrac{5x-8}{(x-1)(x-2)}$.

2. $\dfrac{3x-14}{(x+2)(x-3)}$.

3. $\dfrac{1}{(x-5)(x-6)}$.

4. $\dfrac{x}{(x-5)(x-6)}$.

5. $\dfrac{x+4}{x^2-4}$.

6. $\dfrac{x+4}{x^2-4x}$.

7. $\dfrac{7x-3}{(x-4)(2x-3)}$.

8. $\dfrac{7x+10}{(x+2)(3x+4)}$.

9. $\dfrac{16x-27}{(2x+3)(5x-1)}$.

10. $\dfrac{2x-1}{(5x+2)(2x+5)}$.

11. $\dfrac{9x-28}{x^2-6x+8}$.

12. $\dfrac{3x+2}{x^2+4x-12}$.

13. $\dfrac{11x-12}{x^2-9x+18}$.

14. $\dfrac{9x-29}{6x^2+x-12}$.

15. $\dfrac{6x+37}{12x^2-71x-6}$.

16. $\dfrac{2x+23}{8x^2-26x+15}$.

17. $\dfrac{6x^2-17x+22}{(x+1)(x-2)(x-4)}$.

18. $\dfrac{1}{(x-1)(x-3)(x-5)}$.

19. $\dfrac{7x^2-23x+12}{x^3-5x^2-6x}$.

20. $\dfrac{x^2+21x-18}{x^3-9x}$.

21. $\dfrac{2x-3}{(x-3)^2}$.

22. $\dfrac{6x+11}{(2x+5)^2}$.

23. $\dfrac{4x^2-21x+7}{(x+1)(x-3)^2}$.

24. $\dfrac{x^2+27x+75}{x^3+10x^2+25x}$.

25. $\dfrac{25x}{(x-2)(2x+1)^2}$.

26. $\dfrac{8x^3-16x^2+x+7}{(x+1)^2(x-2)^2}$.

27. $\dfrac{5x}{(x-1)(x^2+4)}$.

28. $\dfrac{3x+4}{(x+2)(x^2+2x+3)}$.

29. $\dfrac{2x^2 - 7}{(2x - 3)(x^2 - 3x + 5)}$.

30. $\dfrac{x^2 + x + 1}{x^3 - x^2 + x}$.

31. $\dfrac{4x^3}{(x^2 - 4)(x^2 + 4)}$.

32. $\dfrac{9x^3}{(x - 2)^2(x^2 + 2)}$.

33. $\dfrac{x}{x^3 + 1}$.

34. $\dfrac{x}{(x + 1)^3}$.

35. $\dfrac{x^3}{(x^2 + 3)^2}$.

36. $\dfrac{x^3 - 1}{(x^2 - x + 1)^2}$.

37. $\dfrac{36}{(x - 2)(x^2 + 2)^2}$.

38. $\dfrac{14x^4 + 7x^3 + 26x^2 + 10x + 3}{(x + 3)(2x^2 - x + 3)^2}$.

39. $\dfrac{108}{(x + 2)^2(x^2 + 2)^2}$.

40. $\dfrac{x^2}{(x^2 + 2)^3}$.

41. $\dfrac{6x^7 - 10x^5 - 4x^4 - 16x^3 - 2x - 4}{(x^2 - 1)^2(x + 1)^2}$.

Reduce each of the following improper fractions to a mixed expression (see section 27) in which the numerator of the fractional part is of lower degree than the denominator. Then separate the fractional part into partial fractions.

42. $\dfrac{x^2 + x + 1}{x^2 + x - 2}$.

43. $\dfrac{x^3 + 7x^2 + 4x - 23}{x^2 + 3x - 10}$.

44. $\dfrac{2x^3 - 3x^2 - 7x + 30}{x^2 - 9}$.

45. $\dfrac{6x^3 - 17x^2 - 31x - 24}{(2x + 1)(x - 4)}$.

46. $\dfrac{x^2 - 9x + 6}{x^2 - 6x + 9}$.

47. $\dfrac{6x^3 + 1}{(3x + 2)(x + 2)^2}$.

48. $\dfrac{8x^4 + 12x^3 - 54x^2 + 38x - 21}{(x + 4)(2x - 1)^2}$.

49. $\dfrac{4x^4 + 6x^3 + 12x^2 + 6x - 3}{(2x + 3)(2x^2 + 3)}$.

50. $\dfrac{x^3 + 1}{x^3 - 1}$.

51. $\dfrac{x^5 + 5}{x^3 - 2x^2 + 3x}$.

CHAPTER XX

Infinite Series

167. Sequences and series.

A **sequence** is a set of numbers arranged in a definite order, for example, the terms of a progression, or the set of positive integers.

A **series** is the indicated sum of a sequence. Thus, if the sequence is $a_1, a_2, a_3, \cdots, a_n, \cdots$, the series is

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots.$$

If a series is the sum of a limited number of terms it is a **finite series,** if it is the indicated sum of an unlimited number of terms it is an **infinite series.** In this chapter we shall consider infinite series.

A series may be defined by means of its general or nth term. For example, if its general term is $a_n = n/(n + 1)$, we find by assigning to n the successive values $1, 2, 3, \cdots$,

$$a_1 = \frac{1}{2}, \quad a_2 = \frac{2}{3}, \quad a_3 = \frac{3}{4}, \cdots.$$

Consequently the series is

$$\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots.$$

On the other hand, a series may be defined by several terms, from which the law of formation of the terms is to be discovered. For example, in the series

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots,$$

we see that $a_1 = \frac{1}{2^1}$, $a_2 = \frac{1}{2^2}$, $a_3 = \frac{1}{2^3}$. Therefore we conclude that $a_n = \frac{1}{2^n}$.

EXERCISES XX. A

Find a formula for the nth term of each of the series in Exercises XX. B, pp. 339–341.

168. Limit.

Consider the unending or infinite sequence

$$S_1, S_2, \cdots, S_n, \cdots.$$

If there is a number S such that, beyond a certain place in the sequence, the absolute value of the difference between S_n and S is smaller than any positive number named in advance, then S is called the **limit** of the sequence. We write

$$\lim_{n \to \infty} S_n = S,$$

which may be read "the limit of S_n, as n increases without limit, is S."

For example, in the infinite geometric progression

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots,$$

let S_n be the sum of the first n terms; we then have

$$S_1 = \frac{1}{2},$$

$$S_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4},$$

$$S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8},$$

$$\cdots\cdots\cdots$$

$$S_n = \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} = 1 - \frac{1}{2^n},$$

$$\cdots\cdots\cdots$$

The limiting value of the sequence of S's is $S = 1$; for the absolute value of the difference between S_n and S, namely $\frac{1}{2^n}$, can be made smaller than any positive number named in advance, by taking n sufficiently large. Thus, if we wish to make this difference less than 0.001 we have merely to take $n = 10$, or any integer greater than 10; for $\frac{1}{2^{10}} = \frac{1}{1024}$, which is less than 0.001. Consequently, any of the terms from S_{10} on differs from 1 by less than 0.001.

169. Convergence and divergence.

In any series,

$$a_1 + a_2 + \cdots + a_{n-1} + a_n + \cdots, \tag{1}$$

let S_n be the sum of the first n terms; that is, let

$$S_1 = a_1,$$
$$S_2 = a_1 + a_2,$$
$$\cdots\cdots$$
$$S_n = a_1 + a_2 + \cdots + a_n,$$
$$\cdots\cdots\cdots$$

If S_n has a limit S the series (1) is said to be **convergent.**

The limit S is called the **sum** of the series. (It is **not** a "sum" in the ordinary sense, but the limit of the sum of n terms as n increases without limit.) A series which is not convergent is said to be **divergent.**

An example of a convergent series is an infinite geometric progression in which the ratio is numerically less than 1.

As an example of divergent series consider an arithmetic progression,

$$2 + 5 + 8 + 11 + \cdots,$$

or a geometric progression whose ratio is greater than 1,

$$1 + 2 + 4 + 8 + \cdots.$$

The sum of n terms of either of these series is seen to increase without limit as we take more and more terms.

170. Necessary condition for convergence.

THEOREM. *If a series is convergent, the nth term approaches zero as n increases without limit.*

Consider the series

$$a_1 + a_2 + \cdots + a_{n-1} + a_n + \cdots$$

with the sum S.

Let S_n be the sum of the first n terms. Then

$$a_n = S_n - S_{n-1}.$$

Thus, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} (S_n - S_{n-1}) = S - S = 0.$ (It is assumed here that the limit of the difference between S_n and S_{n-1} is the difference between their limits.)

Thus, a series is divergent if its general or nth term does not approach zero. On the other hand, the fact that the general term approaches zero does not imply that the series is convergent. In other words, the condition $\lim_{n\to 0} u_n = 0$ is *necessary* but not *sufficient* for convergence.

For example, consider the **harmonic series** (so called because its terms form a harmonic progression),

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} + \cdots. \quad (1)$$

Its general term approaches zero, yet the series is divergent. For the terms may be grouped as follows:

$$1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \left(\frac{1}{9} + \cdots + \frac{1}{16}\right) + \cdots. \quad (2)$$

But

$$\frac{1}{3} + \frac{1}{4} > \frac{1}{4} + \frac{1}{4} = \frac{1}{2},$$

$$\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2},$$

and so on. We can form as many groups of terms as we wish, the sum of each group being greater than $\frac{1}{2}$. By taking enough groups we can make their sum as great as we please. The series therefore diverges, since S_n increases without limit.

171. Fundamental assumption.

If S_n always increases (or always decreases) as n increases, but always remains less (greater) than a fixed value A, then as n increases without limit, S_n approaches a limit which is not greater (less) than A.

172. Comparison test.

The following theorems are often useful in establishing the convergence or divergence of a series by comparing it with a series which is known to be convergent or divergent:

THEOREM I. *If each term of a series of positive terms is less than or equal to the corresponding term of a convergent series, then the given series is convergent.*

Let the given series be

$$a_1 + a_2 + \cdots + a_n + \cdots, \tag{1}$$

and the known convergent series, with sum S', be

$$b_1 + b_2 + \cdots + b_n + \cdots. \tag{2}$$

Let

$$S_n = a_1 + a_2 + \cdots + a_n, \tag{3}$$
$$S'_n = b_1 + b_2 + \cdots + b_n. \tag{4}$$

Since, by hypothesis, $a_k \leqq b_k$ for all values of k, we have

$$S_n \leqq S'_n. \tag{5}$$

But

$$S'_n < S'. \tag{6}$$

It follows from (5) and (6) that

$$S_n < S', \tag{7}$$

and consequently, by the fundamental assumption, the a series (1) is convergent.

THEOREM II. *If each term of a series of positive terms is greater than or equal to the corresponding term of a divergent series of positive terms, then the given series is divergent.*

For, if the given series were convergent the second series would also be, since each of its terms is less than the corresponding term of the first series. But this contradicts the hypothesis that the second series is divergent.

In comparing two series it is not sufficient to compare a few terms at the beginning. The general terms must be compared.

Example 1.

Show that the following series is convergent:

$$1 + \frac{1}{2 \cdot 2} + \frac{1}{3 \cdot 2^2} + \frac{1}{4 \cdot 2^3} + \cdots + \frac{1}{n \cdot 2^{n-1}} + \cdots. \tag{1}$$

SOLUTION. Compare with the convergent geometric series (see section 168),

$$1 + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{n-1}} + \cdots. \tag{2}$$

Each term of (1) is less than or equal to the corresponding term of (2). Hence (1) is convergent.

Example 2.

Show that the following series is divergent:

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} + \cdots. \tag{1}$$

SOLUTION. Each term of (1) is greater than or equal to the corresponding term of the divergent harmonic series (see section 170),

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots.$$

Hence (1) is divergent.

NOTE. *The convergence or divergence of a series is unaffected by omitting (or inserting) a finite number of terms.* For this merely changes S_n by a fixed constant for all values of n. It is often convenient to neglect several terms at the beginning of a series.

173. Useful comparison series.

The following series will be found useful in making comparison tests:

The geometric series,

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots, \tag{1}$$

convergent for $|r| < 1$, divergent for $|r| \geqq 1$.

The p series,

$$1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots, \tag{2}$$

convergent for $p > 1$, divergent for $p \leqq 1$.

If $p = 1$ the series is the divergent harmonic series.

If $p < 1$ the series is greater, term by term, than the harmonic series and is consequently divergent.

If $p > 1$ the convergence of the series can be established as follows:

$$\frac{1}{2^p} + \frac{1}{3^p} < \frac{1}{2^p} + \frac{1}{2^p} = \frac{2}{2^p} = \frac{1}{2^{p-1}},$$

$$\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p} < 4 \cdot \frac{1}{4^p} = \frac{1}{4^{p-1}} = \left(\frac{1}{2^{p-1}}\right)^2,$$

$$\frac{1}{8^p} + \frac{1}{9^p} + \cdots + \frac{1}{15^p} < 8 \cdot \frac{1}{8^p} = \frac{1}{8^{p-1}} = \left(\frac{1}{2^{p-1}}\right)^3,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

Thus,

$$1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots < 1 + \frac{1}{2^{p-1}} + \left(\frac{1}{2^{p-1}}\right)^2 + \cdots. \tag{3}$$

But the expression on the right side of (3) is a convergent geometric series, since its ratio, $1/2^{p-1}$, is less than 1 when $p > 1$. Consequently the p series converges when $p > 1$.

174. Ratio test.

THEOREM. *If, in a series of positive terms, the ratio of the $(n + 1)$th term to the nth term approaches a limit R as n*

increases without limit, then the series is convergent if $R < 1$ *and divergent if* $R > 1$.* *If* $R = 1$ *the series may be convergent or may be divergent.*

Let the series of positive terms be

$$a_1 + a_2 + a_3 + \cdots + a_n + a_{n+1} + \cdots. \tag{1}$$

If $R < 1$, choose a number r between R and 1 (i.e., $R < r < 1$). Since, according to hypothesis,

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = R,$$

then, by the definition of a limit there is a term in the series, say a_k, beyond which a_{n+1}/a_n is less than r. Thus,

$$\frac{a_{k+1}}{a_k} < r, \qquad \text{or} \qquad a_{k+1} < a_k r;$$

$$\frac{a_{k+2}}{a_{k+1}} < r, \qquad \text{or} \qquad a_{k+2} < a_{k+1} r < a_k r^2;$$

$$\frac{a_{k+3}}{a_{k+2}} < r, \qquad \text{or} \qquad a_{k+3} < a_{k+2} r < a_k r^3;$$

$$\cdots\cdots\cdots\cdots\cdots$$

Let us now compare the series

$$a_{k+1} + a_{k+2} + a_{k+3} + \cdots \tag{2}$$

with the series

$$a_k r + a_k r^2 + a_k r^3 + \cdots. \tag{3}$$

Each term of (2) is less than the corresponding term of (3). But (3) is a convergent geometric series, since $r < 1$.

* If the ratio becomes larger without limit (which may be written, symbolically, $R \to \infty$) the series is obviously divergent. This may be included under the case $R > 1$.

Therefore (2) is convergent. However, (2) is the original series (1) with the first k terms omitted, and consequently (1) is convergent.

If $R > 1$, choose a number r between R and 1 (i.e., $R > r > 1$). As before there is a term, say a_k, beyond which we have

$$\frac{a_{k+1}}{a_k} > r, \qquad \text{or} \qquad a_{k+1} > a_k r;$$

$$\frac{a_{k+2}}{a_{k+1}} > r, \qquad \text{or} \qquad a_{k+2} > a_{k+1} r > a_k r^2;$$

$$\frac{a_{k+3}}{a_{k+2}} > r, \qquad \text{or} \qquad a_{k+3} > a_{k+2} r > a_k r^3;$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

Comparing the series

$$a_{k+1} + a_{k+2} + a_{k+3} + \cdots \tag{4}$$

with the geometric series

$$a_k r + a_k r^2 + a_k r^3 + \cdots, \tag{5}$$

which, since $r > 1$, is divergent, we establish the divergence of (4) and consequently of the tested series.

If $R = 1$ we can draw no conclusion from the ratio test, since, as the following illustrations show, there are both convergent and divergent series for which $R = 1$.

Consider the series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} + \cdots. \tag{6}$$

For this series,

$$R = \lim_{n\to\infty} \left(\frac{1}{n+1} \div \frac{1}{n}\right) = \lim_{n\to\infty} \frac{n}{n+1}$$

$$= \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = 1.$$

The ratio test fails to give any information concerning the series (6). It is, however, the divergent harmonic series.

Consider next the series

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} + \cdots. \tag{7}$$

Here

$$R = \lim_{n\to\infty} \left[\frac{1}{(n+1)^2} \div \frac{1}{n^2}\right] = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^2$$

$$= \lim_{n\to\infty} \left(\frac{1}{1+\frac{1}{n}}\right)^2 = 1.$$

The ratio test gives us no information concerning the series (7). However, (7) is the p series with $p > 1$ and is convergent.

Example 1.

Apply the ratio test to the series

$$\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \frac{4}{2^4} + \cdots.$$

SOLUTION.

$$R = \lim_{n\to\infty}\left(\frac{n+1}{2^{n+1}} \div \frac{n}{2^n}\right) = \lim_{n\to\infty}\frac{n+1}{2n}$$

$$= \lim_{n\to\infty}\frac{1+\frac{1}{n}}{2} = \frac{1}{2}.$$

Since $R < 1$, the series is convergent.

Example 2.

Apply the ratio test to the series

$$\frac{2}{1^2} + \frac{2^2}{2^2} + \frac{2^3}{3^2} + \frac{2^4}{4^2} + \cdots.$$

SOLUTION.

$$R = \lim_{n\to\infty}\left[\frac{2^{n+1}}{(n+1)^2} \div \frac{2^n}{n^2}\right] = \lim_{n\to\infty} 2\left(\frac{n}{n+1}\right)^2 = 2.$$

Therefore the series is divergent.

EXERCISES XX. B

Determine whether the following series are convergent or divergent:

1. $\frac{1}{3} + \frac{1}{6} + \frac{1}{9} + \frac{1}{12} + \cdots.$
2. $\frac{1}{3} + \frac{1}{12} + \frac{1}{48} + \frac{1}{192} + \cdots.$
3. $\frac{9}{8} + \frac{3}{4} + \frac{1}{2} + \frac{1}{3} + \cdots.$
4. $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots.$
5. $1 + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \dfrac{1}{4^2} + \cdots.$
6. $\dfrac{1}{1\cdot 3} + \dfrac{1}{3\cdot 5} + \dfrac{1}{5\cdot 7} + \dfrac{1}{7\cdot 9} + \cdots.$
7. $\dfrac{1}{2^2} + \dfrac{2}{3^2} + \dfrac{3}{4^2} + \dfrac{4}{5^2} + \cdots.$

8. $\frac{1}{3}+\frac{2}{3^2}+\frac{3}{3^3}+\frac{4}{3^4}+\cdots$.

9. $\frac{1}{1\cdot 3}+\frac{1}{5\cdot 7}+\frac{1}{9\cdot 11}+\frac{1}{13\cdot 15}+\cdots$.

10. $1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{4}}+\cdots$.

11. $1+\frac{\sqrt[3]{2}}{2}+\frac{\sqrt[3]{3}}{3}+\frac{\sqrt[3]{4}}{4}+\cdots$.

12. $\frac{1}{2\cdot 4}+\frac{1}{4\cdot 6}+\frac{1}{6\cdot 8}+\frac{1}{8\cdot 10}+\cdots$.

13. $1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\cdots$.

14. $\frac{1!}{10}+\frac{2!}{10^2}+\frac{3!}{10^3}+\frac{4!}{10^4}+\cdots$.

15. $\frac{2}{1!}+\frac{2^2}{2!}+\frac{2^3}{3!}+\frac{2^4}{4!}+\cdots$.

16. $\frac{1}{2^2-1}+\frac{1}{3^2-1}+\frac{1}{4^2-1}+\frac{1}{5^2-1}+\cdots$.

17. $\frac{1}{2^2-1}+\frac{1}{3^2-2}+\frac{1}{4^2-3}+\frac{1}{5^2-4}+\cdots$.

18. $\frac{1}{2^2-1}+\frac{1}{2^3-2}+\frac{1}{2^4-3}+\frac{1}{2^5-4}+\cdots$.

19. $\frac{1}{1-0.01}+\frac{1}{1-0.02}+\frac{1}{1-0.03}+\frac{1}{1-0.04}+\cdots$.

20. $\frac{1}{1+0.1}+\frac{1}{1+0.2}+\frac{1}{1+0.3}+\frac{1}{1+0.4}+\cdots$.

21. $\frac{1}{(1+0.1)^2}+\frac{1}{(1+0.2)^2}+\frac{1}{(1+0.3)^2}+\frac{1}{(1+0.4)^2}+\cdots$.

22. $\frac{1}{2}\log_{10}2+\frac{1}{4}\log_{10}4+\frac{1}{8}\log_{10}8+\frac{1}{16}\log_{10}16+\cdots$.

23. $\frac{1}{\log_{10}2}+\frac{1}{\log_{10}3}+\frac{1}{\log_{10}4}+\frac{1}{\log_{10}5}+\cdots$.

24. $\frac{1}{\log_{10} 2} + \frac{1}{\log_{10} 4} + \frac{1}{\log_{10} 8} + \frac{1}{\log_{10} 16} + \cdots.$

25. $\log_{10} \frac{1}{2} + \log_{10} \frac{1}{4} + \log_{10} \frac{1}{8} + \log_{10} \frac{1}{16} + \cdots.$

175. Series with negative terms.

A series with all of its terms negative is not essentially different from a series with all of its terms positive. Thus, if the series

$$a_1 + a_2 + a_3 + \cdots, \tag{1}$$

in which all of the a's are positive, converges to the limit S, then the series

$$-a_1 - a_2 - a_3 - \cdots \tag{2}$$

obviously converges to the limit $-S$. Furthermore, if (1) diverges so also does (2).

The case in which some of the terms of a series are positive and some negative presents a different situation, which will be discussed in the next few sections.

176. Alternating series.

An **alternating series** is one whose terms are alternately positive and negative.

THEOREM. *An alternating series is convergent if the absolute value of each term is less than that of the preceding, and if the limit of the nth term is zero as n increases without limit.*

Let the given series be

$$a_1 - a_2 + a_3 - a_4 + \cdots, \tag{1}$$

in which the a's are positive. The sum of an even number of terms, say $2k$ terms, may be written in either of the following two forms:

$$S_{2k} = (a_1 - a_2) + (a_3 - a_4) + \cdots + (a_{2k-1} - a_{2k}), \tag{2}$$

$$S_{2k} = a_1 - (a_2 - a_3) - \cdots - (a_{2k-2} - a_{2k-1}) - a_{2k}. \tag{3}$$

Since $a_n > a_{n+1}$, each difference in parentheses in (2) and (3) is positive. Then (2) shows that S_{2k} is positive and increases as $2k$ increases, while (3) shows that S_{2k} is never greater than a_1. Therefore, by the fundamental assumption of section 171, S_{2k} approaches a limit S, which is less than or equal to a_1. Equation (3) shows that S is actually less than a_1.

Now consider the sum of an odd number of terms,

$$S_{2k+1} = S_{2k} + a_{2k+1}.$$

We have already shown that S_{2k} approaches a limit S, and by hypothesis a_{2k+1} approaches zero. Therefore S_{2k+1} approaches S. Thus, whether n is odd or even, S_n converges to a limit S.

Example 1.

Show that the following series is convergent:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots.$$

SOLUTION. The series is alternating, and the absolute values of the nth and $(n + 1)$th terms are, respectively,

$$a_n = \frac{1}{n}, \qquad a_{n+1} = \frac{1}{n+1},$$

so that $a_{n+1} < a_n$. Also,

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{1}{n} = 0.$$

Therefore the series is convergent.

COROLLARY. *If the absolute value of each term of an alternating series is less than that of the preceding term, and if the limit of the nth term is zero as n increases without limit, then* *the error made in taking the sum of the first n terms for the sum of the series is less in absolute value than the $(n+1)$th term.*

Given the alternating series,

$$a_1 - a_2 + a_3 - a_4 + \cdots,$$

with the a's positive, $a_{n+1} < a_n$, and $\lim_{n\to\infty} a_n = 0$. If S is the sum of the series, and S_n the sum of the first n terms, we may write

$$\begin{aligned} |S - S_n| &= a_{n+1} - a_{n+2} + a_{n+3} - a_{n+4} + \cdots \\ &= a_{n+1} - (a_{n+2} - a_{n+3}) - (a_{n+4} - a_{n+5}) - \cdots. \end{aligned}$$

Since the differences in parentheses are positive,

$$|S - S_n| < a_{n+1}.$$

This corollary is useful, if we are using an alternating series for purposes of computation, in determining the accuracy of the approximation.

Example 2.

Consider the alternating series,

$$1 - \frac{1}{2} + \frac{1}{2^2} - \cdots + (-1)^{n-1}\frac{1}{2^{n-1}} + (-1)^n\frac{1}{2^n} + \cdots,$$

which satisfies the conditions of the corollary.

The series is an infinite geometric progression whose sum is $1/(1+\frac{1}{2}) = \frac{2}{3}$. The sum of n terms of the progression is

$$\begin{aligned} 1 - \frac{1}{2} + \frac{1}{2^2} - \cdots + (-1)^{n-1}\frac{1}{2^{n-1}} &= 1 \cdot \frac{1 - \left(-\frac{1}{2}\right)^n}{1 + \frac{1}{2}} \\ &= \frac{2}{3}\left[1 - \left(-\frac{1}{2}\right)^n\right]. \end{aligned}$$

The error made in taking the sum of n terms for the sum of the series is

$$\frac{2}{3}\left[1 - \left(-\frac{1}{2}\right)^n\right] - \frac{2}{3} = -\frac{2}{3}\left(-\frac{1}{2}\right)^n = \frac{2}{3}\cdot\frac{(-1)^{n+1}}{2^n}.$$

This error is less in absolute value than the $(n + 1)$th term, $(-1)^n/2^n$.

To illustrate with a definite numerical value of n, let us take $n = 4$. Then the sum of four terms of the series is

$$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} = \frac{5}{8}.$$

Subtracting the true sum of the series, we have for the error,

$$\frac{5}{8} - \frac{2}{3} = -\frac{1}{24}.$$

This is less in absolute value than the 5th term, which is $1/16$.

177. Absolute and conditional convergence.

THEOREM. *A series with both positive and negative terms is convergent if the series of absolute values of its terms is convergent.*

Let the given series be

$$a_1 + a_2 + \cdots + a_n + \cdots, \tag{1}$$

in which some of the a's are positive and some negative. The convergent series of absolute values is, then,

$$|a_1| + |a_2| + \cdots + |a_n| + \cdots. \tag{2}$$

Denote by S_n the sum of the first n terms of (1), by P_n the sum of the positive terms among the first n terms, and by N_n the sum of the absolute values of the negative terms among the first n. Denote by S'_n the sum of the first n terms of (2). Then

$$S_n = P_n - N_n, \qquad S'_n = P_n + N_n. \tag{3}$$

By hypothesis S'_n converges to a limit, say S'. Since (2) is a series of positive terms,

$$S'_n = P_n + N_n < S'$$

for all finite values of n. Hence

$$P_n < S', \qquad N_n < S'.$$

Moreover, since P_n and N_n continually increase with n, by the fundamental assumption of section 171 they approach limits, say P and N respectively. Thus,

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} (P_n - N_n) = P - N,$$

assuming that the limit of a difference is equal to the difference of the limits. That is, series (1) is convergent.

A series is said to be **absolutely convergent** if the series of absolute values of its terms is convergent. If a series with both positive and negative terms is convergent, but the series of absolute values of its terms is divergent, the given series is said to be **conditionally convergent.**

For example, the series

$$1 - \frac{1}{2^2} + \frac{1}{2^3} - \frac{1}{2^4} + \cdots$$

is absolutely convergent, since the series

$$1 + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots$$

is convergent.

The series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

is conditionally convergent. We proved in section 176 that it is convergent, but the series of absolute values of its terms is

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots,$$

the divergent harmonic series.

178. Ratio test extended.

THEOREM. *If in a series of real terms the absolute value of the ratio of the $(n + 1)$th term to the nth term approaches a limit R as n increases without limit, then the series is absolutely convergent if $R < 1$ and divergent if $R > 1$. If $R = 1$ the series may be convergent or may be divergent.*

Since R is the limit of the ratio of the $(n + 1)$th to the nth term of the series of absolute values, by the Theorem of section 174, this series of absolute values is convergent if $R < 1$. That is to say, the given series is absolutely convergent if $R < 1$.

If $R > 1$, each term, after a certain place in the series, will be numerically greater than the preceding, and consequently the general term does not approach zero. Therefore the series diverges. (See section 170.)

If $R = 1$ no conclusion can be drawn.

EXERCISES XX. C

Determine whether the following series are convergent or divergent:

1. $1 - \frac{1}{\sqrt[3]{2}} + \frac{1}{\sqrt[3]{3}} - \frac{1}{\sqrt[3]{4}} + \cdots.$

2. $\frac{1}{4} - \frac{3}{6} + \frac{5}{8} - \frac{7}{10} + \cdots$.
3. $\frac{1}{2} - \frac{2}{3} + \frac{3}{4} - \frac{4}{5} + \cdots$.
4. $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$.
5. $\frac{1}{2^2} - \frac{2}{3^2} + \frac{3}{4^2} - \frac{4}{5^2} + \cdots$.
6. $1 - \frac{1}{1 + 0.1} + \frac{1}{1 + 0.2} - \frac{1}{1 + 0.3} + \cdots$.
7. $\frac{1 - 10}{2} - \frac{2 - 10}{3} + \frac{3 - 10}{4} - \frac{4 - 10}{5} + \cdots$.
8. $\log_{10} \sqrt{2} - \log_{10} \sqrt[3]{2} + \log_{10} \sqrt[4]{2} - \log_{10} \sqrt[5]{2} + \cdots$.
9. $\log_{10} \frac{1}{2} - \log_{10} \frac{1}{3} + \log_{10} \frac{1}{4} - \log_{10} \frac{1}{5} + \cdots$.
10. $\log_{10} \frac{1}{2} - \log_{10} \frac{1}{4} + \log_{10} \frac{1}{8} - \log_{10} \frac{1}{16} + \cdots$.
11. $\frac{1}{1 + 0.1} - \frac{1}{1 + 0.2} + \frac{1}{1 + 0.3} - \frac{1}{1 + 0.4} + \cdots$.
12. $\frac{1}{1 - 0.1} - \frac{1}{1 - 0.02} + \frac{1}{1 - 0.003} - \frac{1}{1 - 0.0004} + \cdots$.
13. $(1 - 0.1)^2 - (1 - 0.01)^2 + (1 - 0.02)^2 - (1 - 0.03)^2 + \cdots$.
14. $2^{-1/2} - 3^{-1/3} + 4^{-1/4} - 5^{-1/5} + \cdots$.
15. $\frac{1}{2 - \sqrt{2}} - \frac{1}{3 - \sqrt{3}} + \frac{1}{4 - \sqrt{4}} - \frac{1}{5 - \sqrt{5}} + \cdots$.
16. $\frac{\sqrt{1}}{2 - \sqrt{3}} - \frac{\sqrt{2}}{3 - \sqrt{4}} + \frac{\sqrt{3}}{4 - \sqrt{5}} - \frac{\sqrt{4}}{5 - \sqrt{6}} + \cdots$.
17. $\frac{2}{2 \cdot 3} - \frac{2^2}{3 \cdot 4} + \frac{2^3}{4 \cdot 5} - \frac{2^4}{5 \cdot 6} + \cdots$.
18. $\frac{1!}{100} - \frac{2!}{(100)^2} + \frac{3!}{(100)^3} - \frac{4!}{(100)^4} + \cdots$.

179. Power series.

A series of the form

$$c_0 + c_1x + c_2x^2 + \cdots + c_nx^n + \cdots, \tag{1}$$

in which the c's are constants, is called **a power series in x.**

A series such as

$$c_0 + c_1(x - a) + c_2(x - a)^2 + \cdots + c_n(x - a)^n + \cdots, \quad (2)$$

is a power series in $x - a$.

The set of values of x for which a power series converges is called the **interval of convergence.** This interval of convergence can be determined by means of the extended form of the ratio test.

Example.

Find the interval of convergence of the following series:

$$x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots.$$

SOLUTION.

$$R = \lim_{n \to \infty} \frac{x^{n+1}}{n+1} \div \frac{x^n}{n} = \lim_{n \to \infty} \frac{n}{n+1} |x| = |x|.$$

The series is convergent if $R < 1$, that is, if $|x| < 1$.

The end values, $x = \pm 1$, must be investigated separately. For $x = 1$ the series is the divergent harmonic series, for $x = -1$ it is a convergent alternating series. Thus, the interval of convergence is

$$-1 \leqq x < 1.$$

EXERCISES XX. D

Find the interval of convergence of each of the following series:

1. $\frac{x}{2} + \frac{x^2}{4} + \frac{x^3}{6} + \frac{x^4}{8} + \cdots.$
2. $1 + \frac{x}{3} + \frac{x^2}{9} + \frac{x^3}{27} + \cdots.$
3. $1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots.$
4. $\frac{x}{1^2} + \frac{x^2}{2^2} + \frac{x^3}{3^2} + \frac{x^4}{4^2} + \cdots.$
5. $1 + x^2 + x^4 + x^6 + \cdots.$
6. $1 - x^2 + x^4 - x^6 + \cdots.$

7. $\frac{1}{1\cdot 2}+\frac{x}{2\cdot 3}+\frac{x^2}{3\cdot 4}+\frac{x^3}{4\cdot 5}+\cdots.$

8. $\frac{1}{1\cdot 2}+\frac{x}{3\cdot 4}+\frac{x^2}{5\cdot 6}+\frac{x^3}{7\cdot 8}+\cdots.$

9. $\frac{x}{1\cdot 2^2}+\frac{x^2}{2\cdot 3^2}+\frac{x^3}{3\cdot 4^2}+\frac{x^4}{4\cdot 5^2}+\cdots.$

10. $x-x^2\sqrt{2}+x^3\sqrt{3}-x^4\sqrt{4}+\cdots.$

11. $x+2x^2+3x^3+4x^4+\cdots.$

12. $1+5x+25x^2+125x^3+\cdots.$

13. $1+\frac{x-1}{2}+\frac{(x-1)^2}{4}+\frac{(x-1)^3}{8}+\cdots.$

14. $(x+2)+\frac{(x+2)^2}{2}+\frac{(x+2)^3}{3}+\frac{(x+2)^4}{4}+\cdots.$

15. $\frac{1}{1\cdot 3}+\frac{x}{5\cdot 7}+\frac{x^2}{9\cdot 11}+\frac{x^3}{13\cdot 15}+\cdots.$

16. $1+\frac{\frac{1}{2}x-1}{3}+\frac{(\frac{1}{2}x-1)^2}{9}+\frac{(\frac{1}{2}x-1)^3}{27}+\cdots.$

17. $x+2!x^2+3!x^3+4!x^4+\cdots.$

18. $1+(x-a)+\frac{(x-a)^2}{2!}+\frac{(x-a)^3}{3!}+\cdots.$

19. $1+\frac{x-a}{b}+\frac{(x-a)^2}{b^2}+\frac{(x-a)^3}{b^3}+\cdots.$

20. $\frac{x-a}{b}+\frac{(x-a)^2}{2b}+\frac{(x-a)^3}{3b}+\frac{(x-a)^4}{4b}+\cdots.$

21. $\frac{1\cdot 2}{3\cdot 4}x+\frac{3\cdot 4}{5\cdot 6}x^2+\frac{5\cdot 6}{7\cdot 8}x^3+\frac{7\cdot 8}{9\cdot 10}x^4+\cdots.$

CHAPTER XXI

Finite Differences

180. Differences.

The subject of **finite differences** deals with the changes in a function caused by finite changes in the independent variable.

Let f be a function of an independent variable which increases by regular steps:

$$x, x + h, x + 2h, x + 3h, \cdots, x + mh, \cdots.$$

We indicate the differences between successive values of the function by the following notation:

$$\begin{aligned}
\Delta f(x) &= f(x + h) - f(x), \\
\Delta f(x + h) &= f(x + 2h) - f(x + h), \\
\Delta f(x + 2h) &= f(x + 3h) - f(x + 2h), \\
&\cdots\cdots\cdots\cdots \\
\Delta f(x + mh) &= f[x + (m + 1)h] - f(x + mh), \\
&\cdots\cdots\cdots\cdots
\end{aligned}$$

The symbol $\Delta f(x)$ is read "delta of f of x." Note that Δ is not a number, but is a symbol for the operation of subtracting, or **differencing;** thus, $\Delta f(x)$ does *not* mean "Δ times $f(x)$."

The above Δ's are called **first differences** of the function f. We may similarly form the differences of the Δ's, which are called the **second differences,** or **differences of order two,** of the function f. The notation is as follows:

$$\begin{aligned}\Delta^2 f(x) &= \Delta[\Delta f(x)] = \Delta[f(x+h) - f(x)] \\ &= f(x+2h) - f(x+h) - [f(x+h) - f(x)] \\ &= f(x+2h) - 2f(x+h) + f(x).\end{aligned}$$

The symbol $\Delta^2 f(x)$ may be read "delta second of f of x." Similarly,

$$\begin{aligned}\Delta^2 f(x+h) &= f(x+3h) - 2f(x+2h) + f(x+h), \\ \Delta^2 f(x+2h) &= f(x+4h) - 2f(x+3h) + f(x+2h), \\ &\cdots\cdots\cdots\cdots\end{aligned}$$

The differences of these second differences are called **third differences,** $\Delta^3 f(x)$, and so on.

The successive differences of a function may be conveniently arranged in a table such as the accompanying one. Note that the difference between two values is placed in the next column and on a line between the two values. (Sometimes the differences of $f(x)$ are placed on the same line as $f(x)$, the differences of $f(x+h)$ on the same line as $f(x+h)$, and so on, but the arrangement given in the accompanying table is to be preferred.)

Function	First differences	Second differences	Third differences
$f(x)$			
	$\Delta f(x)$		
$f(x+h)$		$\Delta^2 f(x)$	
	$\Delta f(x+h)$		$\Delta^3 f(x)$
$f(x+2h)$		$\Delta^2 f(x+h)$	
	$\Delta f(x+2h)$		$\Delta^3 f(x+h)$
$f(x+3h)$		$\Delta^2 f(x+2h)$	
	$\Delta f(x+3h)$		
$f(x+4h)$			
$\cdots$			

A numerical example will make the process of differencing clearer. For simplicity we take $x = 0$ and $h = 1$; then

$x + mh = m$. Suppose that our function is $f(x + mh) = f(m) = m^3$. We form the table of differences as shown.

Table of Differences of m^3

m	$f(m)$	Δ	Δ^2	Δ^3
0	0			
		1		
1	1		6	
		7		6
2	8		12	
		19		6
3	27		18	
		37		6
4	64		24	
		61		
5	125			

Note that all the third differences which we have found are constant. It can be proved that all third differences of this function are constant, and that consequently the fourth (and all higher-order) differences are zero. It can be proved in general that *for an integral rational function (that is, a polynomial) of degree n, the nth differences are constant,* and *conversely,* that *if all the nth differences are constant, the function is a polynomial of degree n.**

181. Finding any term of a numerical series.

One use of differences is that of finding the terms of a series of numbers when the law of formation of the series is not given, but when it is known that the differences of a certain order are constant. To derive an appropriate formula for this use we must work backwards from the differences to the function. Thus, recalling how the various differences were obtained, we see that

* See Hall and Knight's algebras.

$$
\begin{aligned}
f(x+h) &= f(x) + \Delta f(x),\\
f(x+2h) &= f(x+h) + \Delta f(x+h)\\
&= [f(x) + \Delta f(x)] + [\Delta f(x) + \Delta^2 f(x)]\\
&= f(x) + 2\Delta f(x) + \Delta^2 f(x),\\
f(x+3h) &= f(x+2h) + \Delta f(x+2h)\\
&= [f(x) + 2\Delta f(x) + \Delta^2 f(x)]\\
&\qquad + [\Delta f(x+h) + \Delta^2 f(x+h)]\\
&= [f(x) + 2\Delta f(x) + \Delta^2 f(x)] + [\Delta f(x) + \Delta^2 f(x)]\\
&\qquad + [\Delta^2 f(x) + \Delta^3 f(x)]\\
&= f(x) + 3\Delta f(x) + 3\Delta^2 f(x) + \Delta^3 f(x).
\end{aligned}
$$

The coefficients are those of the binomial formula, and it can be proved by mathematical induction that, for positive integral values of m, whatever the function $f(x)$,

$$
f(x+mh) = f(x) + m\Delta f(x) + \frac{m(m-1)}{1\cdot 2}\Delta^2 f(x) + \frac{m(m-1)(m-2)}{1\cdot 2\cdot 3}\Delta^3 f(x) + \cdots + \Delta^m f(x). \quad (1)
$$

If m is not a positive integer, (1) becomes

$$
f(x+mh) = f(x) + m\Delta f(x) + \frac{m(m-1)}{1\cdot 2}\Delta^2 f(x) + \cdots, \quad (2)
$$

the right side of which will not terminate unless $f(x)$ is a polynomial. When $f(x)$ is a polynomial of degree n, differences of order n are constant, and the right side will terminate with the term in $\Delta^n f(x)$. In this case it can be further proved that (2) is valid for all rational values of m, both positive and negative.

Many functions can be approximately represented by polynomials, or in other words they have differences of some order which are nearly equal. Formula (2) can be successfully applied to such functions, and consequently has a wide range of practical applicability.

Its use in finding any term of a numerical series will now be illustrated.

Example.

Find (a) $f(8)$, (b) $f(m)$, in the series whose terms are

$$5, 3, 1, 5, 21, 55, \cdots.$$

Solution. Here $x = 0, h = 1$. Form the table of differences. If we can assume that all third differences are constant we may proceed as follows:

m	$f(m)$	Δ	Δ^2	Δ^3
0	5			
		−2		
1	3		0	
		−2		6
2	1		6	
		4		6
3	5		12	
		16		6
4	21		18	
		34		
5	55			

(a) Set $m = 8$ in (1):

$$f(8) = 5 + 8(-2) + \frac{8 \cdot 7}{1 \cdot 2} \cdot 0 + \frac{8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3} \cdot 6$$

$$= 5 - 16 + 336 = 325.$$

(Also see note after solution of part (b).)

(b) $$f(m) = 5 + m(-2) + \frac{m(m-1)}{1 \cdot 2} \cdot 0 + \frac{m(m-1)(m-2)}{1 \cdot 2 \cdot 3} \cdot 6$$

$$= 5 - 2m + m(m-1)(m-2) = m^3 - 3m^2 + 5.$$

It can readily be verified that the values of $f(m)$ listed in the second column of the table can be obtained by substituting the appropriate values of m in the expression $m^3 - 3m^2 + 5$.

Note that the value of $f(8)$ can be obtained without the direct use of formula (1) by building up the table of differences. Thus, $18 + 6 = 24$, which is placed below 18 in the Δ^2 column. Next, $34 + 24 = 58$, placed below 34 in the Δ column; $55 + 58 = 113$, placed below 55 in the $f(m)$ column.

m	$f(m)$	Δ	Δ^2	Δ^3
			18	
		34		6
5	55		24	
		58		6
6	113		30	
		88		6
7	201		36	
		124		
8	325			

Starting at the right again, we get $24 + 6 = 30$ (placed below 24 in the Δ^2 column), and so on.

The process can be continued until $f(8) = 325$ is obtained.

182. Interpolation.

One extremely important application of formula (2) of the preceding section is in interpolation. This will be illustrated by a numerical example.

Suppose that we wish to find log 1.325 from a table of logarithms given for numbers at intervals of one-tenth. We find log 1.3, log 1.4, etc., and form the accompanying table of differences. In the differences, decimal points are omitted (e.g., the difference tabulated as 322 is actually 0.0322). Note that the third differences are small and practically equal. In such a case further differencing is useless.

m	$x + mh$	$\log(x + mh)$	Δ	Δ^2	Δ^3
0	1.3	0.1139			
			322		
1	1.4	0.1461		−22	
			300		2
2	1.5	0.1761		−20	
			280		3
3	1.6	0.2041		−17	
			263		3
4	1.7	0.2304		−14	
			249		
5	1.8	0.2553			

In this illustration, $x = 1.3$, $h = 0.1$, and the value of m which will make $x + mh$ equal to 1.325 is 0.25. Substituting these values in (2) of section 181, we find

$$\log 1.325 = 0.1139 + 0.25 \times 0.0322$$
$$+\frac{0.25(-0.75)}{1 \cdot 2}(-0.0022) + \frac{0.25(-0.75)(-1.75)}{1 \cdot 2 \cdot 3} \times 0.0002$$
$$= 0.1139 + 0.00805 + 0.00020625 + 0.000011 = 0.1222.$$

Note that the term involving the third difference has no effect on the fourth place of the final result.

The process of interpolation given in Chapter XIV is equivalent to using only the first two terms of formula (2) of section 181.

183. Summation of series.

Still another use of finite differences is in summing numerical series. Let F be a function whose difference is f; that is,

$$F(x + h) - F(x) = f(x),$$
$$F(x + 2h) - F(x + h) = f(x + h),$$
$$F(x + 3h) - F(x + 2h) = f(x + 2h),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$F(x + mh) - F(x + \overline{m - 1}\, h) = f(x + \overline{m - 1}\, h).$$

Adding the foregoing equations, we get

$$F(x + mh) - F(x) = f(x) + f(x + h) + \cdots + f(x + \overline{m - 1}\,h). \quad (1)$$

If use is made of the relations $\Delta F(x) = f(x)$, $\Delta^2 F(x) = \Delta f(x)$, etc., formula (1) of section 181 may be written in the form

$$F(x + mh) = F(x) + mf(x) + \frac{m(m-1)}{1 \cdot 2}\Delta f(x) + \frac{m(m-1)(m-2)}{1 \cdot 2 \cdot 3}\Delta^2 f(x) + \cdots + \Delta^{m-1} f(x). \quad (2)$$

Hence, from (1) and (2), we get

$$f(x) + f(x + h) + \cdots + f(x + \overline{m - 1}\,h) = mf(x) + \frac{m(m-1)}{1 \cdot 2}\Delta f(x) + \frac{m(m-1)(m-2)}{1 \cdot 2 \cdot 3}\Delta^2 f(x) + \cdots + \Delta^{m-1} f(x). \quad (3)$$

Example.

Find an expression for the sum of the first m terms of the series whose terms are

$$5, 3, 1, 5, 21, 55, \cdots.$$

SOLUTION. $x = 0, h = 1$; $f(0) = 5, f(1) = 3, \cdots$. The sum of the first m terms is

$$f(0) + f(1) + \cdots + f(m - 1),$$

which by formula (3) above is

m	$f(m)$	$\Delta f(m)$	$\Delta^2 f(m)$	$\Delta^3 f(m)$
0	5			
		−2		
1	3		0	
		−2		6
2	1		6	
		4		6
3	5		12	
		16		6
4	21		18	
		34		
5	55			

$$mf(0) + \frac{m(m-1)}{1\cdot 2}\Delta f(0) + \cdots$$

$$= m\cdot 5 + \frac{m(m-1)}{1\cdot 2}(-2) + 0 + \frac{m(m-1)(m-2)(m-3)}{1\cdot 2\cdot 3\cdot 4}\cdot 6$$

$$= \frac{m}{4}(m^3 - 6m^2 + 7m + 18).$$

This can readily be checked for a particular value of m. For example,

$$F(3) = \frac{3}{4}(27 - 54 + 21 + 18) = 9.$$

The sum of the first 3 terms of the series is $5 + 3 + 1 = 9$.

EXERCISES XXI. A

Difference the series whose terms are:

1. 2, 5, 10, 17, 26, · · ·.
2. 3, 9, 13, 15, 15, · · ·.
3. 1, 4, 15, 40, 85, 156, · · ·.
4. −54, −42, −10, 38, 95, 151, 193, · · ·.
5. −10, −12, −6, 38, 174, 480, 1058, · · ·.

6. Find the 8th term and a formula for the mth term of the series in exercise 1.
7. Find the 12th term and a formula for the mth term of the series in exercise 2.

NOTE. The mth term is not $f(m)$ in the notation of section 181.

8. Find the 10th term and a formula for the mth term of the series in exercise 3.
9. Find the 9th term and a formula for the mth term of the series in exercise 4.
10. Find the 10th term and a formula for the mth term of the series in exercise 5.
11. Given $\log_{10} 1.5 = 0.1761$, $\log_{10} 1.6 = 0.2041$, $\log_{10} 1.7 = 0.2304$, $\log_{10} 1.8 = 0.2553$, $\log_{10} 1.9 = 0.2788$. Find $\log_{10} 1.536$.
12. Given $\log_e 1.2 = 0.1823$, $\log_e 1.3 = 0.2624$, $\log_e 1.4 = 0.3365$. Find $\log_e 1.284$.
13. Given $e = 2.7183$, $e^{1.1} = 3.0042$, $e^{1.2} = 3.3201$, $e^{1.3} = 3.6693$, $e^{1.4} = 4.0552$. Find $e^{1.075}$.
14. Using the accompanying table, find sine, cosine, and tangent of each of the following angles: (a) $40° 19' 12''$, (b) $40° 45' 36''$, (c) $40° 10' 48''$.

	40°	41°	42°	43°
sin	0.64279	0.65606	0.66913	0.68200
cos	0.76604	0.75471	0.74314	0.73135
tan	0.83910	0.86929	0.90040	0.93252

15. Find the sum of 10 terms and a formula for the sum of m terms of the series in exercises 1–5.

Find expressions for

16. $1^2 + 2^2 + 3^2 + \cdots + m^2$.
17. $1^3 + 2^3 + 3^3 + \cdots + m^3$.
18. $1^4 + 2^4 + 3^4 + \cdots + m^4$.

Find expressions for the sum of m terms of the series:

19. $1^2 + 3^2 + 5^2 + 7^2 + \cdots$.

20. $1^3 + 3^3 + 5^3 + 7^3 + \cdots$.

Find the sum of m terms of the series whose nth term is

21. $n^3 - 2n^2 + 3n$.

22. $n^3 - 3n + 2$.

23. $2n^3 - 3n^2 + 4n - 5$.

24. $n^4 - 3n^3 - 2n^2 - 4n$.

TABLES

INDEX

ANSWERS

Table I.—Powers and Roots

No.	Square	Cube	Square root	Cube root	No.	Square	Cube	Square root	Cube root
1	1	1	1.000	1.000	51	2 601	132 651	7.141	3.708
2	4	8	1.414	1.260	52	2 704	140 608	7.211	3.733
3	9	27	1.732	1.442	53	2 809	148 877	7.280	3.756
4	16	64	2.000	1.587	54	2 916	157 464	7.348	3.780
5	25	125	2.236	1.710	55	3 025	166 375	7.416	3.803
6	36	216	2.449	1.817	56	3 136	175 616	7.483	3.826
7	49	343	2.646	1.913	57	3 249	185 193	7.550	3.849
8	64	512	2.828	2.000	58	3 364	195 112	7.616	3.871
9	81	729	3.000	2.080	59	3 481	205 379	7.681	3.893
10	100	1 000	3.162	2.154	**60**	3 600	216 000	7.746	3.915
11	121	1 331	3.317	2.224	61	3 721	226 981	7.810	3.936
12	144	1 728	3.464	2.289	62	3 844	238 328	7.874	3.958
13	169	2 197	3.606	2.351	63	3 969	250 047	7.937	3.979
14	196	2 744	3.742	2.410	64	4 096	262 144	8.000	4.000
15	225	3 375	3.873	2.466	65	4 225	274 625	8.062	4.021
16	256	4 096	4.000	2.520	66	4 356	287 496	8.124	4.041
17	289	4 913	4.123	2.571	67	4 489	300 763	8.185	4.062
18	324	5 832	4.243	2.621	68	4 624	314 432	8.246	4.082
19	361	6 859	4.359	2.668	69	4 761	328 509	8.307	4.102
20	400	8 000	4.472	2.714	**70**	4 900	343 000	8.367	4.121
21	441	9 261	4.583	2.759	71	5 041	357 911	8.426	4.141
22	484	10 648	4.690	2.802	72	5 184	373 248	8.485	4.160
23	529	12 167	4.796	2.844	73	5 329	389 017	8.544	4.179
24	576	13 824	4.899	2.884	74	5 476	405 224	8.602	4.198
25	625	15 625	5.000	2.924	75	5 625	421 875	8.660	4.217
26	676	17 576	5.099	2.962	76	5 776	438 976	8.718	4.236
27	729	19 683	5.196	3.000	77	5 929	456 533	8.775	4.254
28	784	21 952	5.292	3.037	78	6 084	474 552	8.832	4.273
29	841	24 389	5.385	3.072	79	6 241	493 039	8.888	4.291
30	900	27 000	5.477	3.107	**80**	6 400	512 000	8.944	4.309
31	961	29 791	5.568	3.141	81	6 561	531 441	9.000	4.327
32	1 024	32 768	5.657	3.175	82	6 724	551 368	9.055	4.344
33	1 089	35 937	5.745	3.208	83	6 889	571 787	9.110	4.362
34	1 156	39 304	5.831	3.240	84	7 056	592 704	9.165	4.380
35	1 225	42 875	5.916	3.271	85	7 225	614 125	9.220	4.397
36	1 296	46 656	6.000	3.302	86	7 396	636 056	9.274	4.414
37	1 369	50 653	6.083	3.332	87	7 569	658 503	9.327	4.431
38	1 444	54 872	6.164	3.362	88	7 744	681 472	9.381	4.448
39	1 521	59 319	6.245	3.391	89	7 921	704 969	9.434	4.465
40	1 600	64 000	6.325	3.420	**90**	8 100	729 000	9.487	4.481
41	1 681	68 921	6.403	3.448	91	8 281	753 571	9.539	4.498
42	1 764	74 088	6.481	3.476	92	8 464	778 688	9.592	4.514
43	1 849	79 507	6.557	3.503	93	8 649	804 357	9.644	4.531
44	1 936	85 184	6.633	3.530	94	8 836	830 584	9.695	4.547
45	2 025	91 125	6.708	3.557	95	9 025	857 375	9.747	4.563
46	2 116	97 336	6.782	3.583	96	9 216	884 736	9.798	4.579
47	2 209	103 823	6.856	3.609	97	9 409	912 673	9.849	4.595
48	2 304	110 592	6.928	3.634	98	9 604	941 192	9.899	4.610
49	2 401	117 649	7.000	3.659	99	9 801	970 299	9.950	4.626
50	2 500	125 000	7.071	3.684	**100**	10 000	1 000 000	10.000	4.642

N	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396

N	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996

TABLE III.—COMPOUND AMOUNT: $(1 + r)^n$

n	1%	1½%	2%	2½%	3%	4%	5%	6%	7%
1	1.0100	1.0150	1.0200	1.0250	1.0300	1.0400	1.0500	1.0600	1.0700
2	1.0201	1.0302	1.0404	1.0506	1.0609	1.0816	1.1025	1.1236	1.1449
3	1.0303	1.0457	1.0612	1.0769	1.0927	1.1249	1.1576	1.1910	1.2250
4	1.0406	1.0614	1.0824	1.1038	1.1255	1.1699	1.2155	1.2625	1.3108
5	1.0510	1.0773	1.1041	1.1314	1.1593	1.2167	1.2763	1.3382	1.4026
6	1.0615	1.0934	1.1262	1.1597	1.1941	1.2653	1.3401	1.4185	1.5007
7	1.0721	1.1098	1.1487	1.1887	1.2299	1.3159	1.4071	1.5036	1.6058
8	1.0829	1.1265	1.1717	1.2184	1.2668	1.3686	1.4775	1.5938	1.7182
9	1.0937	1.1434	1.1951	1.2489	1.3048	1.4233	1.5513	1.6895	1.8385
10	1.1046	1.1605	1.2190	1.2801	1.3439	1.4802	1.6289	1.7908	1.9672
11	1.1157	1.1779	1.2434	1.3121	1.3842	1.5395	1.7103	1.8983	2.1049
12	1.1268	1.1956	1.2682	1.3449	1.4258	1.6010	1.7959	2.0122	2.2522
13	1.1381	1.2136	1.2936	1.3785	1.4685	1.6651	1.8856	2.1329	2.4098
14	1.1495	1.2318	1.3195	1.4130	1.5126	1.7317	1.9799	2.2609	2.5785
15	1.1610	1.2502	1.3459	1.4483	1.5580	1.8009	2.0789	2.3966	2.7590
16	1.1726	1.2690	1.3728	1.4845	1.6047	1.8730	2.1829	2.5404	2.9522
17	1.1843	1.2880	1.4002	1.5216	1.6528	1.9479	2.2920	2.6928	3.1588
18	1.1961	1.3073	1.4282	1.5597	1.7024	2.0258	2.4066	2.8543	3.3799
19	1.2081	1.3270	1.4568	1.5987	1.7535	2.1068	2.5270	3.0256	3.6165
20	1.2202	1.3469	1.4859	1.6386	1.8061	2.1911	2.6533	3.2071	3.8697
21	1.2324	1.3671	1.5157	1.6796	1.8603	2.2788	2.7860	3.3996	4.1406
22	1.2447	1.3876	1.5460	1.7216	1.9161	2.3699	2.9253	3.6035	4.4304
23	1.2572	1.4084	1.5769	1.7646	1.9736	2.4647	3.0715	3.8197	4.7405
24	1.2697	1.4295	1.6084	1.8087	2.0328	2.5633	3.2251	4.0489	5.0724
25	1.2824	1.4509	1.6406	1.8539	2.0938	2.6658	3.3864	4.2919	5.4274
26	1.2953	1.4727	1.6734	1.9003	2.1566	2.7725	3.5557	4.5494	5.8074
27	1.3082	1.4948	1.7069	1.9478	2.2213	2.8834	3.7335	4.8223	6.2139
28	1.3213	1.5172	1.7410	1.9965	2.2879	2.9987	3.9201	5.1117	6.6488
29	1.3345	1.5400	1.7758	2.0464	2.3566	3.1187	4.1161	5.4184	7.1143
30	1.3478	1.5631	1.8114	2.0976	2.4273	3.2434	4.3219	5.7435	7.6123
31	1.3613	1.5865	1.8476	2.1500	2.5001	3.3731	4.5380	6.0881	8.1451
32	1.3749	1.6103	1.8845	2.2038	2.5751	3.5081	4.7649	6.4534	8.7153
33	1.3887	1.6345	1.9222	2.2589	2.6523	3.6484	5.0032	6.8406	9.3253
34	1.4026	1.6590	1.9607	2.3153	2.7319	3.7943	5.2533	7.2510	9.9781
35	1.4166	1.6839	1.9999	2.3732	2.8139	3.9461	5.5160	7.6861	10.6766
36	1.4308	1.7091	2.0399	2.4325	2.8983	4.1039	5.7918	8.1473	11.4239
37	1.4451	1.7348	2.0807	2.4933	2.9852	4.2681	6.0814	8.6361	12.2236
38	1.4595	1.7608	2.1223	2.5557	3.0748	4.4388	6.3855	9.1543	13.0793
39	1.4741	1.7872	2.1647	2.6196	3.1670	4.6164	6.7048	9.7035	13.9948
40	1.4889	1.8140	2.2080	2.6851	3.2620	4.8010	7.0400	10.2857	14.9745
41	1.5038	1.8412	2.2522	2.7522	3.3599	4.9931	7.3920	10.9029	16.0227
42	1.5188	1.8688	2.2972	2.8210	3.4607	5.1928	7.7616	11.5570	17.1443
43	1.5340	1.8969	2.3432	2.8915	3.5645	5.4005	8.1497	12.2505	18.3444
44	1.5493	1.9253	2.3901	2.9638	3.6715	5.6165	8.5572	12.9855	19.6285
45	1.5648	1.9542	2.4379	3.0379	3.7816	5.8412	8.9850	13.7646	21.0025
46	1.5805	1.9835	2.4866	3.1139	3.8950	6.0748	9.4343	14.5905	22.4726
47	1.5963	2.0133	2.5363	3.1917	4.0119	6.3178	9.9060	15.4659	24.0457
48	1.6122	2.0435	2.5871	3.2715	4.1323	6.5705	10.4013	16.3939	25.7289
49	1.6283	2.0741	2.6388	3.3533	4.2562	6.8333	10.9213	17.3775	27.5299
50	1.6446	2.1052	2.6916	3.4371	4.3839	7.1067	11.4674	18.4202	29.4570

Table IV.—Present Value: $(1 + r)^{-n}$

n	1%	1½%	2%	2½%	3%	4%	5%	6%	7%
1	.99010	.98522	.98039	.97561	.97087	.96154	.95238	.94340	.93458
2	98030	97066	.96117	.95181	.94260	.92456	.90703	.89000	.87344
3	.97059	95632	94232	.92860	.91514	.88900	.86384	.83962	.81630
4	.96098	.94218	.92385	.90595	.88849	.85480	.82270	.79209	.76290
5	.95147	.92826	.90573	.88385	.86261	.82193	.78353	.74726	.71299
6	.94205	.91454	.88797	.86230	.83748	.79031	.74622	.70496	.66634
7	.93272	.90103	.87056	.84127	.81309	.75992	.71068	.66506	.62275
8	.92348	.88771	.85349	.82075	.78941	.73069	.67684	.62741	.58201
9	91434	.87459	.83676	.80073	.76642	.70259	.64461	.59190	.54393
10	90529	.86167	.82035	.78120	.74409	.67556	.61391	.55839	.50835
11	.89632	.84893	.80426	.76214	.72242	.64958	.58468	.52679	.47509
12	.88745	.83639	.78849	.74356	.70138	.62460	.55684	.49697	.44401
13	.87866	.82403	77303	.72542	.68095	.60057	.53032	.46884	.41496
14	.86996	.81185	.75788	.70773	.66112	.57748	.50507	.44230	.38782
15	.86135	.79985	.74301	.69047	.64186	.55526	.48102	.41727	.36245
16	.85282	.78803	.72845	.67362	.62317	.53391	.45811	.39365	.33873
17	.84438	.77639	.71416	.65720	.60502	.51337	.43630	.37136	.31657
18	.83602	.76491	.70016	.64117	.58739	.49363	.41552	.35034	.29586
19	.82774	.75361	.68643	.62553	.57029	.47464	.39573	.33051	.27651
20	.81954	.74247	.67297	.61027	.55368	.45639	.37689	.31180	.25842
21	.81143	.73150	.65978	.59539	.53755	.43883	.35894	.29416	.24151
22	.80340	.72069	.64684	.58086	.52189	.42196	.34185	.27751	.22571
23	.79544	.71004	.63416	.56670	.50669	.40573	.32557	.26180	.21095
24	.78757	.69954	.62172	.55288	.49193	.39012	.31007	.24698	.19715
25	.77977	.68921	.60953	.53939	.47761	.37512	.29530	.23300	.18425
26	.77205	.67902	.59758	.52623	.46369	.36069	.28124	.21981	.17220
27	.76440	.66899	.58586	.51340	.45019	.34682	.26785	.20737	.16093
28	.75684	.65910	.57437	.50088	.43708	.33348	.25509	.19563	.15040
29	.74934	.64936	.56311	.48866	.42435	.32065	.24295	.18456	.14056
30	.74192	.63976	.55207	.47674	.41199	.30832	.23138	.17411	.13137
31	.73458	.63031	.54125	.46511	.39999	.29646	.22036	.16425	.12277
32	.72730	.62099	.53063	.45377	.38834	.28506	.20987	.15496	.11474
33	.72010	.61182	.52023	.44270	.37703	.27409	.19987	.14619	.10723
34	.71297	.60277	.51003	.43191	.36604	.26355	.19035	.13791	.10022
35	.70591	.59387	.50003	.42137	.35538	.25342	.18129	.13011	.09366
36	.69892	.58509	.49022	.41109	.34503	.24367	.17266	.12274	.08754
37	.69200	.57644	.48061	.40107	.33498	.23430	.16444	.11579	.08181
38	.68515	.56792	.47119	.39128	.32523	.22529	.15661	.10924	.07646
39	.67837	.55953	.46195	.38174	.31575	.21662	.14915	.10306	.07146
40	.67165	.55126	.45289	.37243	.30656	.20829	.14205	.09722	.06678
41	.66500	.54312	.44401	.36335	.29763	.20028	.13528	.09172	.06241
42	.65842	.53509	.43530	.35448	.28896	.19257	.12884	.08653	.05833
43	.65190	.52718	.42677	.34584	.28054	.18517	.12270	.08163	.05451
44	.64545	.51939	.41840	.33740	.27237	.17805	.11686	.07701	.05095
45	.63905	.51171	.41020	.32917	.26444	.17120	.11130	.07265	.04761
46	.63273	.50415	.40215	.32115	.25674	.16461	.10600	.06854	.04450
47	.62646	.49670	.39427	.31331	.24926	.15828	.10095	.06466	.04159
48	.62026	.48936	.38654	.30567	.24200	.15219	.09614	.06100	.03887
49	.61412	.48213	.37896	.29822	.23495	.14634	.09156	.05755	.03632
50	.60804	.47500	.37153	.29094	.22811	.14071	.08720	.05429	.03395

TABLE V.—AMOUNT OF AN ANNUITY: $[(1 + r)^n - 1]/r$

n	1%	1½%	2%	2½%	3%	4%	5%	6%	7%
1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
2	2.0100	2.0150	2.0200	2.0250	2.0300	2.0400	2.0500	2.0600	2.0700
3	3.0301	3.0452	3.0604	3.0756	3.0909	3.1216	3.1525	3.1836	3.2149
4	4.0604	4.0909	4.1216	4.1525	4.1836	4.2465	4.3101	4.3746	4.4399
5	5.1010	5.1523	5.2040	5.2563	5.3091	5.4163	5.5256	5.6371	5.7507
6	6.1520	6.2296	6.3081	6.3877	6.4684	6.6330	6.8019	6.9753	7.1533
7	7.2135	7.3230	7.4343	7.5474	7.6625	7.8983	8.1420	8.3938	8.6540
8	8.2857	8.4328	8.5830	8.7361	8.8923	9.2142	9.5491	9.8975	10.2598
9	9.3685	9.5593	9.7546	9.9545	10.1591	10.5828	11.0266	11.4913	11.9780
10	10.4622	10.7027	10.9497	11.2034	11.4639	12.0061	12.5779	13.1808	13.8164
11	11.5668	11.8633	12.1687	12.4835	12.8078	13.4864	14.2068	14.9716	15.7836
12	12.6825	13.0412	13.4121	13.7956	14.1920	15.0258	15.9171	16.8699	17.8885
13	13.8093	14.2368	14.6803	15.1404	15.6178	16.6268	17.7130	18.8821	20.1406
14	14.9474	15.4504	15.9739	16.5190	17.0863	18.2919	19.5986	21.0151	22.5505
15	16.0969	16.6821	17.2934	17.9319	18.5989	20.0236	21.5786	23.2760	25.1290
16	17.2579	17.9324	18.6393	19.3802	20.1569	21.8245	23.6575	25.6725	27.8881
17	18.4304	19.2014	20.0121	20.8647	21.7616	23.6975	25.8404	28.2129	30.8402
18	19.6147	20.4894	21.4123	22.3863	23.4144	25.6454	28.1324	30.9057	33.9990
19	20.8109	21.7967	22.8406	23.9460	25.1169	27.6712	30.5390	33.7600	37.3790
20	22.01 0	23.1237	24.2974	25.5447	26.8704	29.7781	33.0660	36.7856	40.9955
21	23.2392	24.4705	25.7833	27.1833	28.6765	31.9692	35.7193	39.9927	44.8652
22	24.4716	25.8376	27.2990	28.8629	30.5368	34.2480	38.5052	43.3923	49.0057
23	25.7163	27.2251	28.8450	30.5844	32.4529	36.6179	41.4305	46.9958	53.4361
24	26.9735	28.6335	30.4219	32.3490	34.4265	39.0826	44.5020	50.8156	58.1767
25	28.2432	30.0630	32.0303	34.1578	36.4593	41.6459	47.7271	54.8645	63.2490
26	29.5256	31.5140	33.6709	36.0117	38.5530	44.3117	51.1135	59.1564	68.6765
27	30.8209	32.9867	35.3443	37.9120	40.7096	47.0842	54.6691	63.7058	74.4838
28	32.1291	34.4815	37.0512	39.8598	42.9309	49.9676	58.4026	68.5281	80.6977
29	33.4504	35.9987	38.7922	41.8563	45.2189	52.9663	62.3227	73.6398	87.3465
30	34.7849	37.5387	40.5681	43.9027	47.5754	56.0849	66.4388	79.0582	94.4608
31	36.1327	39.1018	42.3794	46.0003	50.0027	59.3283	70.7608	84.8017	102.0730
32	37.4941	40.6883	44.2270	48.1503	52.5028	62.7015	75.2988	90.8898	110.2182
33	38.8690	42.2986	46.1116	50.3540	55.0778	66.2095	80.0638	97.3432	118.9334
34	40.2577	43.9331	48.0338	52.6129	57.7302	69.8579	85.0670	104.1838	128.2588
35	41.6603	45.5921	49.9945	54.9282	60.4621	73.6522	90.3203	111.4348	138.2369
36	43.0769	47.2760	51.9944	57.3014	63.2759	77.5983	95.8363	119.1209	148.9135
37	44.5076	48.9851	54.0343	59.7339	66.1742	81.7022	101.6281	127.2681	160.3374
38	45.9527	50.7199	56.1149	62.2273	69.1594	85.9703	107.7095	135.9042	172.5610
39	47.4123	52.4807	58.2372	64.7830	72.2342	90.4091	114.0950	145.0585	185.6403
40	48.8864	54.2679	60.4020	67.4026	75.4013	95.0255	120.7998	154.7620	199.6351
41	50.3752	56.0819	62.6100	70.0876	78.6633	99.8265	127.8398	165.0477	214.6096
42	51.8790	57.9231	64.8622	72.8398	82.0232	104.8196	135.2318	175.9505	230.6322
43	53.3978	59.7920	67.1595	75.6608	85.4839	110.0124	142.9933	187.5076	247.7765
44	54.9318	61.6889	69.5027	78.5523	89.0484	115.4129	151.1430	199.7580	266.1209
45	56.4811	63.6142	71.8927	81.5161	92.7199	121.0294	159.7002	212.7435	285.7493
46	58.0459	65.5684	74.3306	84.5540	96.5015	126.8706	168.6852	226.5081	306.7518
47	59.6263	67.5519	76.8172	87.6679	100.3965	132.9454	178.1194	241.0986	329.2244
48	61.2226	69.5652	79.3535	90.8596	104.4084	139.2632	188.0254	256.5645	353.2701
49	62.8348	71.6087	81.9406	94.1311	108.5406	145.8337	198.4267	272.9584	378.9990
50	64.4632	73.6828	84.5794	97.4843	112.7969	152.6671	209.3480	290.3359	406.5289

TABLE VI.—PRESENT VALUE OF AN ANNUITY: $[1 - (1 + r)^{-n}]/r$

n	1%	1½%	2%	2½%	3%	4%	5%	6%	7%
1	.9901	.9852	.9804	.9756	.9709	.9615	.9524	.9434	.9346
2	1.9704	1.9559	1.9416	1.9274	1.9135	1.8861	1.8594	1.8334	1.8080
3	2.9410	2.9122	2.8839	2.8560	2.8286	2.7751	2.7232	2.6730	2.6243
4	3.9020	3.8544	3.8077	3.7620	3.7171	3.6299	3.5460	3.4651	3.3872
5	4.8534	4.7826	4.7135	4.6458	4.5797	4.4518	4.3295	4.2124	4.1002
6	5.7955	5.6972	5.6014	5.5081	5.4172	5.2421	5.0757	4.9173	4.7665
7	6.7282	6.5982	6.4720	6.3494	6.2303	6.0021	5.7864	5.5824	5.3893
8	7.6517	7.4859	7.3255	7.1701	7.0197	6.7327	6.4632	6.2098	5.9713
9	8.5660	8.3605	8.1622	7.9709	7.7861	7.4353	7.1078	6.8017	6.5152
10	9.4713	9.2222	8.9826	8.7521	8.5302	8.1109	7.7217	7.3601	7.0236
11	10.3676	10.0711	9.7868	9.5142	9.2526	8.7605	8.3064	7.8869	7.4987
12	11.2551	10.9075	10.5753	10.2578	9.9540	9.3851	8.8633	8.3838	7.9427
13	12.1337	11.7315	11.3484	10.9832	10.6350	9.9856	9.3936	8.8527	8.3577
14	13.0037	12.5434	12.1062	11.6909	11.2961	10.5631	9.8986	9.2950	8.7455
15	13.8651	13.3432	12.8493	12.3814	11.9379	11.1184	10.3797	9.7122	9.1079
16	14.7179	14.1313	13.5777	13.0550	12.5611	11.6523	10.8378	10.1059	9.4466
17	15.5623	14.9076	14.2919	13.7122	13.1661	12.1657	11.2741	10.4773	9.7632
18	16.3983	15.6726	14.9920	14.3534	13.7535	12.6593	11.6896	10.8276	10.0591
19	17.2260	16.4262	15.6785	14.9789	14.3238	13.1339	12.0853	11.1581	10.3356
20	18.0456	17.1686	16.3514	15.5892	14.8775	13.5903	12.4622	11.4699	10.5940
21	18.8570	17.9001	17.0112	16.1845	15.4150	14.0292	12.8212	11.7641	10.8355
22	19.6604	18.6208	17.6580	16.7654	15.9369	14.4511	13.1630	12.0416	11.0612
23	20.4558	19.3309	18.2922	17.3321	16.4436	14.8568	13.4886	12.3034	11.2722
24	21.2434	20.0304	18.9139	17.8850	16.9355	15.2470	13.7986	12.5504	11.4693
25	22.0232	20.7196	19.5235	18.4244	17.4131	15.6221	14.0939	12.7834	11.6536
26	22.7952	21.3986	20.1210	18.9506	17.8768	15.9828	14.3752	13.0032	11.8258
27	23.5596	22.0676	20.7069	19.4640	18.3270	16.3296	14.6430	13.2105	11.9867
28	24.3164	22.7267	21.2813	19.9649	18.7641	16.6631	14.8981	13.4062	12.1371
29	25.0658	23.3761	21.8444	20.4535	19.1885	16.9837	15.1411	13.5907	12.2777
30	25.8077	24.0158	22.3965	20.9303	19.6004	17.2920	15.3725	13.7648	12.4090
31	26.5423	24.6461	22.9377	21.3954	20.0004	17.5885	15.5928	13.9291	12.5318
32	27.2696	25.2671	23.4683	21.8492	20.3888	17.8736	15.8027	14.0840	12.6466
33	27.9897	25.8790	23.9886	22.2919	20.7658	18.1476	16.0025	14.2302	12.7538
34	28.7027	26.4817	24.4986	22.7238	21.1318	18.4112	16.1929	14.3681	12.8540
35	29.4086	27.0756	24.9986	23.1452	21.4872	18.6646	16.3742	14.4982	12.9477
36	30.1075	27.6607	25.4888	23.5563	21.8323	18.9083	16.5469	14.6210	13.0352
37	30.7995	28.2371	25.9695	23.9573	22.1672	19.1426	16.7113	14.7368	13.1170
38	31.4847	28.8051	26.4406	24.3486	22.4925	19.3679	16.8679	14.8460	13.1935
39	32.1630	29.3646	26.9026	24.7303	22.8082	19.5845	17.0170	14.9491	13.2649
40	32.8347	29.9158	27.3555	25.1028	23.1148	19.7928	17.1591	15.0463	13.3317
41	33.4997	30.4590	27.7995	25.4661	23.4124	19.9931	17.2944	15.1380	13.3941
42	34.1581	30.9941	28.2348	25.8206	23.7014	20.1856	17.4232	15.2245	13.4524
43	34.8100	31.5212	28.6616	26.1664	23.9819	20.3708	17.5459	15.3062	13.5070
44	35.4555	32.0406	29.0800	26.5038	24.2543	20.5488	17.6628	15.3832	13.5579
45	36.0945	32.5523	29.4902	26.8330	24.5187	20.7200	17.7741	15.4558	13.6055
46	36.7272	33.0565	29.8923	27.1542	24.7754	20.8847	17.8801	15.5244	13.6500
47	37.3537	33.5532	30.2866	27.4675	25.0247	21.0429	17.9810	15.5890	13.6916
48	37.9740	34.0426	30.6731	27.7732	25.2667	21.1951	18.0772	15.6500	13.7305
49	38.5881	34.5247	31.0521	28.0714	25.5017	21.3415	18.1687	15.7076	13.7668
50	39.1961	34.9997	31.4236	28.3623	25.7298	21.4822	18.2559	15.7619	13.8007

TABLE VII.—AMERICAN EXPERIENCE TABLE OF MORTALITY

Age	Number living	Number dying	Age	Number living	Number dying	Age	Number living	Number dying
10	100,000	749	40	78,106	765	70	38,569	2,391
11	99,251	746	41	77,341	774	71	36,178	2,448
12	98,505	743	42	76,567	785	72	33,730	2,487
13	97,762	740	43	75,782	797	73	31,243	2,505
14	97,022	737	44	74,985	812	74	28,738	2,501
15	96,285	735	45	74,173	828	75	26,237	2,476
16	95,550	732	46	73,345	848	76	23,761	2,431
17	94,818	729	47	72,497	870	77	21,330	2,369
18	94,089	727	48	71,627	896	78	18,961	2,291
19	93,362	725	49	70,731	927	79	16,670	2,196
20	92,637	723	50	69,804	962	80	14,474	2,091
21	91,914	722	51	68,842	1,001	81	12,383	1,964
22	91,192	721	52	67,841	1,044	82	10,419	1,816
23	90,471	720	53	66,797	1,091	83	8,603	1,648
24	89,751	719	54	65,706	1,143	84	6,955	1,470
25	89,032	718	55	64,563	1,199	85	5,485	1,292
26	88,314	718	56	63,364	1,260	86	4,193	1,114
27	87,596	718	57	62,104	1,325	87	3,079	933
28	86,878	718	58	60,779	1,394	88	2,146	744
29	86,160	719	59	59,385	1,468	89	1,402	555
30	85,441	720	60	57,917	1,546	90	847	385
31	84,721	721	61	56,371	1,628	91	462	246
32	84,000	723	62	54,743	1,713	92	216	137
33	83,277	726	63	53,030	1,800	93	79	58
34	82,551	729	64	51,230	1,889	94	21	18
35	81,822	732	65	49,341	1,980	95	3	3
36	81,090	737	66	47,361	2,070			
37	80,353	742	67	45,291	2,158			
38	79,611	749	68	43,133	2,243			
39	78,862	756	69	40,890	2,321			

angle	sin	tan	cot	cos	
0° 00′	.0000	.0000	—	1.0000	90° 00′
10	.0029	.0029	343.77	1.0000	50
20	.0058	.0058	171.89	1.0000	40
30	.0087	.0087	114.59	1.0000	30
40	.0116	.0116	85.940	.9999	20
50	.0145	.0145	68.750	.9999	10
1° 00′	.0175	.0175	57.290	.9998	89° 00′
10	.0204	.0204	49.104	.9998	50
20	.0233	.0233	42.964	.9997	40
30	.0262	.0262	38.188	.9997	30
40	.0291	.0291	34.368	.9996	20
50	.0320	.0320	31.242	.9995	10
2° 00′	.0349	.0349	28.636	.9994	88° 00′
10	.0378	.0378	26.432	.9993	50
20	.0407	.0407	24.542	.9992	40
30	.0436	.0437	22.904	.9990	30
40	.0465	.0466	21.470	.9989	20
50	.0494	.0495	20.206	.9988	10
3° 00′	.0523	.0524	19.081	.9986	87° 00′
10	.0552	.0553	18.075	.9985	50
20	.0581	.0582	17.169	.9983	40
30	.0610	.0612	16.350	.9981	30
40	.0640	.0641	15.605	.9980	20
50	.0669	.0670	14.924	.9978	10
4° 00′	.0698	.0699	14.301	.9976	86° 00′
10	.0727	.0729	13.727	.9974	50
20	.0756	.0758	13.197	.9971	40
30	.0785	.0787	12.706	.9969	30
40	.0814	.0816	12.251	.9967	20
50	.0843	.0846	11.826	.9964	10
5° 00′	.0872	.0875	11.430	.9962	85° 00′
10	.0901	.0904	11.059	.9959	50
20	.0929	.0934	10.712	.9957	40
30	.0958	.0963	10.385	.9954	30
40	.0987	.0992	10.078	.9951	20
50	.1016	.1022	9.7882	.9948	10
6° 00′	.1045	.1051	9.5144	.9945	84° 00′
10	.1074	.1080	9.2553	.9942	50
20	.1103	.1110	9.0098	.9939	40
30	.1132	.1139	8.7769	.9936	30
40	.1161	.1169	8.5555	.9932	20
50	.1190	.1198	8.3450	.9929	10
7° 00′	.1219	.1228	8.1443	.9925	83° 00′
10	.1248	.1257	7.9530	.9922	50
20	.1276	.1287	7.7704	.9918	40
30	.1305	.1317	7.5958	.9914	30
40	.1334	.1346	7.4287	.9911	20
50	.1363	.1376	7.2687	.9907	10
8° 00′	.1392	.1405	7.1154	.9903	82° 00′
10	.1421	.1435	6.9682	.9899	50
20	.1449	.1465	6.8269	.9894	40
30	.1478	.1495	6.6912	.9890	30
40	.1507	.1524	6.5606	.9886	20
50	.1536	.1554	6.4348	.9881	10
9° 00′	.1564	.1584	6.3138	.9877	81° 00′
	cos	cot	tan	sin	angle

angle	sin	tan	cot	cos	
9° 00′	.1564	.1584	6.3138	.9877	81° 00′
10	.1593	.1614	6.1970	.9872	50
20	.1622	.1644	6.0844	.9868	40
30	.1650	.1673	5.9758	.9863	30
40	.1679	.1703	5.8708	.9858	20
50	.1708	.1733	5.7694	.9853	10
10° 00′	.1736	.1763	5.6713	.9848	80° 00′
10	.1765	.1793	5.5764	.9843	50
20	.1794	.1823	5.4845	.9838	40
30	.1822	.1853	5.3955	.9833	30
40	.1851	.1883	5.3093	.9827	20
50	.1880	.1914	5.2257	.9822	10
11° 00′	.1908	.1944	5.1446	.9816	79° 00′
10	.1937	.1974	5.0658	.9811	50
20	.1965	.2004	4.9894	.9805	40
30	.1994	.2035	4.9152	.9799	30
40	.2022	.2065	4.8430	.9793	20
50	.2051	.2095	4.7729	.9787	10
12° 00′	.2079	.2126	4.7046	.9781	78° 00′
10	.2108	.2156	4.6382	.9775	50
20	.2136	.2186	4.5736	.9769	40
30	.2164	.2217	4.5107	.9763	30
40	.2193	.2247	4.4494	.9757	20
50	.2221	.2278	4.3897	.9750	10
13° 00′	.2250	.2309	4.3315	.9744	77° 00′
10	.2278	.2339	4.2747	.9737	50
20	.2306	.2370	4.2193	.9730	40
30	.2334	.2401	4.1653	.9724	30
40	.2363	.2432	4.1126	.9717	20
50	.2391	.2462	4.0611	.9710	10
14° 00′	.2419	.2493	4.0108	.9703	76° 00′
10	.2447	.2524	3.9617	.9696	50
20	.2476	.2555	3.9136	.9689	40
30	.2504	.2586	3.8667	.9681	30
40	.2532	.2617	3.8208	.9674	20
50	.2560	.2648	3.7760	.9667	10
15° 00′	.2588	.2679	3.7321	.9659	75° 00′
10	.2616	.2711	3.6891	.9652	50
20	.2644	.2742	3.6470	.9644	40
30	.2672	.2773	3.6059	.9636	30
40	.2700	.2805	3.5656	.9628	20
50	.2728	.2836	3.5261	.9621	10
16° 00′	.2756	.2867	3.4874	.9613	74° 00′
10	.2784	.2899	3.4495	.9605	50
20	.2812	.2931	3.4124	.9596	40
30	.2840	.2962	3.3759	.9588	30
40	.2868	.2994	3.3402	.9580	20
50	.2896	.3026	3.3052	.9572	10
17° 00′	.2924	.3057	3.2709	.9563	73° 00′
10	.2952	.3089	3.2371	.9555	50
20	.2979	.3121	3.2041	.9546	40
30	.3007	.3153	3.1716	.9537	30
40	.3035	.3185	3.1397	.9528	20
50	.3062	.3217	3.1084	.9520	10
18° 00′	.3090	.3249	3.0777	.9511	72° 00′
	cos	cot	tan	sin	angle

Table VIII. — Trigonometric Functions (*Continued*)

angle	sin	tan	cot	cos	
18° 00′	.3090	.3249	3.0777	.9511	**72° 00′**
10	.3118	.3281	3.0475	.9502	50
20	.3145	.3314	3.0178	.9492	40
30	.3173	.3346	2.9887	.9483	30
40	.3201	.3378	2.9600	.9474	20
50	.3228	.3411	2.9319	.9465	10
19° 00′	.3256	.3443	2.9042	.9455	**71° 00′**
10	.3283	.3476	2.8770	.9446	50
20	.3311	.3508	2.8502	.9436	40
30	.3338	.3541	2.8239	.9426	30
40	.3365	.3574	2.7980	.9417	20
50	.3393	.3607	2.7725	.9407	10
20° 00′	.3420	.3640	2.7475	.9397	**70° 00′**
10	.3448	.3673	2.7228	.9387	50
20	.3475	.3706	2.6985	.9377	40
30	.3502	.3739	2.6746	.9367	30
40	.3529	.3772	2.6511	.9356	20
50	.3557	.3805	2.6279	.9346	10
21° 00′	.3584	.3839	2.6051	.9336	**69° 00′**
10	.3611	.3872	2.5826	.9325	50
20	.3638	.3906	2.5605	.9315	40
30	.3665	.3939	2.5386	.9304	30
40	.3692	.3973	2.5172	.9293	20
50	.3719	.4006	2.4960	.9283	10
22° 00′	.3746	.4040	2.4751	.9272	**68° 00′**
10	.3773	.4074	2.4545	.9261	50
20	.3800	.4108	2.4342	.9250	40
30	.3827	.4142	2.4142	.9239	30
40	.3854	.4176	2.3945	.9228	20
50	.3881	.4210	2.3750	.9216	10
23° 00′	.3907	.4245	2.3559	.9205	**67° 00′**
10	.3934	.4279	2.3369	.9194	50
20	.3961	.4314	2.3183	.9182	40
30	.3987	.4348	2.2998	.9171	30
40	.4014	.4383	2.2817	.9159	20
50	.4041	.4417	2.2637	.9147	10
24° 00′	.4067	.4452	2.2460	.9135	**66° 00′**
10	.4094	.4487	2.2286	.9124	50
20	.4120	.4522	2.2113	.9112	40
30	.4147	.4557	2.1943	.9100	30
40	.4173	.4592	2.1775	.9088	20
50	.4200	.4628	2.1609	.9075	10
25° 00′	.4226	.4663	2.1445	.9063	**65° 00′**
10	.4253	.4699	2.1283	.9051	50
20	.4279	.4734	2.1123	.9038	40
30	.4305	.4770	2.0965	.9026	30
40	.4331	.4806	2.0809	.9013	20
50	.4358	.4841	2.0655	.9001	10
26° 00′	.4384	.4877	2.0503	.8988	**64° 00′**
10	.4410	.4913	2.0353	.8975	50
20	.4436	.4950	2.0204	.8962	40
30	.4462	.4986	2.0057	.8949	30
40	.4488	.5022	1.9912	.8936	20
50	.4514	.5059	1.9768	.8923	10
27° 00′	.4540	.5095	1.9626	.8910	**63° 00′**
	cos	cot	tan	sin	angle

angle	sin	tan	cot	cos	
27° 00′	.4540	.5095	1.9626	.8910	**63° 00′**
10	.4566	.5132	1.9486	.8897	50
20	.4592	.5169	1.9347	.8884	40
30	.4617	.5206	1.9210	.8870	30
40	.4643	.5243	1.9074	.8857	20
50	.4669	.5280	1.8940	.8843	10
28° 00′	.4695	.5317	1.8807	.8829	**62° 00′**
10	.4720	.5354	1.8676	.8816	50
20	.4746	.5392	1.8546	.8802	40
30	.4772	.5430	1.8418	.8788	30
40	.4797	.5467	1.8291	.8774	20
50	.4823	.5505	1.8165	.8760	10
29° 00′	.4848	.5543	1.8040	.8746	**61° 00′**
10	.4874	.5581	1.7917	.8732	50
20	.4899	.5619	1.7796	.8718	40
30	.4924	.5658	1.7675	.8704	30
40	.4950	.5696	1.7556	.8689	20
50	.4975	.5735	1.7437	.8675	10
30° 00′	.5000	.5774	1.7321	.8660	**60° 00′**
10	.5025	.5812	1.7205	.8646	50
20	.5050	.5851	1.7090	.8631	40
30	.5075	.5890	1.6977	.8616	30
40	.5100	.5930	1.6864	.8601	20
50	.5125	.5969	1.6753	.8587	10
31° 00′	.5150	.6009	1.6643	.8572	**59° 00′**
10	.5175	.6048	1.6534	.8557	50
20	.5200	.6088	1.6426	.8542	40
30	.5225	.6128	1.6319	.8526	30
40	.5250	.6168	1.6212	.8511	20
50	.5275	.6208	1.6107	.8496	10
32° 00′	.5299	.6249	1.6003	.8480	**58° 00′**
10	.5324	.6289	1.5900	.8465	50
20	.5348	.6330	1.5798	.8450	40
30	.5373	.6371	1.5697	.8434	30
40	.5398	.6412	1.5597	.8418	20
50	.5422	.6453	1.5497	.8403	10
33° 00′	.5446	.6494	1.5399	.8387	**57° 00′**
10	.5471	.6536	1.5301	.8371	50
20	.5495	.6577	1.5204	.8355	40
30	.5519	.6619	1.5108	.8339	30
40	.5544	.6661	1.5013	.8323	20
50	.5568	.6703	1.4919	.8307	10
34° 00′	.5592	.6745	1.4826	.8290	**56° 00′**
10	.5616	.6787	1.4733	.8274	50
20	.5640	.6830	1.4641	.8258	40
30	.5664	.6873	1.4550	.8241	30
40	.5688	.6916	1.4460	.8225	20
50	.5712	.6959	1.4370	.8208	10
35° 00′	.5736	.7002	1.4281	.8192	**55° 00′**
10	.5760	.7046	1.4193	.8175	50
20	.5783	.7089	1.4106	.8158	40
30	.5807	.7133	1.4019	.8141	30
40	.5831	.7177	1.3934	.8124	20
50	.5854	.7221	1.3848	.8107	10
36° 00′	.5878	.7265	1.3764	.8090	**54° 00′**
	cos	cot	tan	sin	angle

angle	sin	tan	cot	cos	
36° 00′	.5878	.7265	1.3764	.8090	**54°** 00′
10	.5901	.7310	1.3680	.8073	50
20	.5925	.7355	1.3597	.8056	40
30	.5948	.7400	1.3514	.8039	30
40	.5972	.7445	1.3432	.8021	20
50	.5995	.7490	1.3351	.8004	10
37° 00′	.6018	.7536	1.3270	.7986	**53°** 00′
10	.6041	.7581	1.3190	.7969	50
20	.6065	.7627	1.3111	.7951	40
30	.6088	.7673	1.3032	.7934	30
40	.6111	.7720	1.2954	.7916	20
50	.6134	.7766	1.2876	.7898	10
38° 00′	.6157	.7813	1.2799	.7880	**52°** 00′
10	.6180	.7860	1.2723	.7862	50
20	.6202	.7907	1.2647	.7844	40
30	.6225	.7954	1.2572	.7826	30
40	.6248	.8002	1.2497	.7808	20
50	.6271	.8050	1.2423	.7790	10
39° 00′	.6293	.8098	1.2349	.7771	**51°** 00′
10	.6316	.8146	1.2276	.7753	50
20	.6338	.8195	1.2203	.7735	40
30	.6361	.8243	1.2131	.7716	30
40	.6383	.8292	1.2059	.7698	20
50	.6406	.8342	1.1988	.7679	10
40° 00′	.6428	.8391	1.1918	.7660	**50°** 00′
10	.6450	.8441	1.1847	.7642	50
20	.6472	.8491	1.1778	.7623	40
30	.6494	.8541	1.1708	.7604	30
40	.6517	.8591	1.1640	.7585	20
50	.6539	.8642	1.1571	.7566	10
41° 00′	.6561	.8693	1.1504	.7547	**49°** 00′
10	.6583	.8744	1.1436	.7528	50
20	.6604	.8796	1.1369	.7509	40
30	.6626	.8847	1.1303	.7490	30
40	.6648	.8899	1.1237	.7470	20
50	.6670	.8952	1.1171	.7451	10
42° 00′	.6691	.9004	1.1106	.7431	**48°** 00′
10	.6713	.9057	1.1041	.7412	50
20	.6734	.9110	1.0977	.7392	40
30	.6756	.9163	1.0913	.7373	30
40	.6777	.9217	1.0850	.7353	20
50	.6799	.9271	1.0786	.7333	10
43° 00′	.6820	.9325	1.0724	.7314	**47°** 00′
10	.6841	.9380	1.0661	.7294	50
20	.6862	.9435	1.0599	.7274	40
30	.6884	.9490	1.0538	.7254	30
40	.6905	.9545	1.0477	.7234	20
50	.6926	.9601	1.0416	.7214	10
44° 00′	.6947	.9657	1.0355	.7193	**46°** 00′
10	.6967	.9713	1.0295	.7173	50
20	.6988	.9770	1.0235	.7153	40
30	.7009	.9827	1.0176	.7133	30
40	.7030	.9884	1.0117	.7112	20
50	.7050	.9942	1.0058	.7092	10
45° 00′	.7071	1.0000	1.0000	.7071	**45°** 00′
	cos	cot	tan	sin	angle

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Answers to Odd-Numbered Exercises

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1. 3.
3. $4a + 2b - c - ab + ac$.
5. $96p + 64q - 24r + 24s$.
7. $2a - 38b - 30$.
9. $z - (x + 2y + 3)$.
11. $a - (-2b + 3c + 4d)$.

Exercises I. B, page 8

1. a^8.
3. x^7y^7.
5. $12p^7$.
7. $9a^6$.
9. $-16x^4$.
11. $-64x^6$.
13. $3x^6$.
15. $\dfrac{7x^2z^4}{2}$.
17. $\dfrac{1}{m^6}$.
19. $-\dfrac{3r^5}{2t^2}$.
21. a^9.
23. a^7.
25. $24x^8$.
27. $6x^4$.

Exercises I. C, page 10

1. $2x^3 + 3x^2 - 2x + 21$.
3. $12a^3 + 16a^2 - 41a - 15$.
5. $4n^4 - 16m^3 - 12m^2 + 24m + 9$.
7. $2x^4 + x^3y - 7x^2y^2 + 19xy^3 - 15y^4$.
9. $2x^5 - 5x^4 + 2x^3 + 2x^2 - 13x + 6$.
11. $14a^6 - 13a^5 + 18a^4 - 29a^3 - 40a^2 + 50a - 8$.
13. $6x^2 - 6y^2 + 4z^2 - 5w^2 + 5xy - 11xz - 13xw - 10yz + 13yw + 19zw$.
15. $4a^6 + 12a^3b^2 + 9b^4$.
17. $a^2 + b^2 + c^2 + 2ab - 2ac - 2bc$.
19. $2x^6 + x^5 + 10x^3 - 20x^2 + 21x - 6$.
21. $x^3 - 3x^2y + 3xy^2 - y^3$.
23. $36a^4 - 156a^3b + 241a^2b^2 - 156ab^3 + 36b^4$.

Exercises I. D, page 12

The first expression given is the quotient, the second is the remainder. If the latter is 0 it is not given.

1. $x^2 - 3x - 2$.
3. $2x + 3$.
5. $-3x^2 + 2x - 4$.
7. $4a^2 - 2a + 15$.

9. $3x + 2y$.
11. $x^2 - 5x - 5, -12$.
13. $4x - 2, -15x + 10$.
15. $x^3 + 5x^2 + 23x + 15, 75$.
17. $x + y, 2y^2$.
19. $x^2 + xy + y^2, 2y^3$.
21. $x^4 - x^3y + x^2y^2 - xy^3 + y^4$.
23. $a^3 + 2a^2b + 2ab^2 + b^3$.

Exercises I. E, page 14

1. $2, 5, 8, -1, -4, 3a^2 + 2, -3a + 2, 5 - 3y$.
3. 4, 2, 14, 44, 9704.
5. $6, -15, -32, -\frac{4}{3}, -\frac{25}{3}, 8a^4 - 20a^2 + 12$.
7. $4a^2 - 1$.
9. $\frac{3}{5}, -\frac{9}{8}, \frac{a}{2}, 3b, \frac{2a}{3}$.
11. $A = f(d) = \frac{\pi d^2}{4}$.
13. $A = f(c) = \frac{c^2}{4\pi}$.
15. $d = f(t) = 60t$.
17. $L = f(r,h) = 2\pi rh$.
19. $h = f(V,r) = \frac{3V}{\pi r^2}$.

Exercises I. G, page 19

1. $4a - 4b$.
3. $62a - 30b - 32$.
5. $28x + 6y + 12z - 9xy + 9xz$.
7. $24a^5b^7c^3$.
9. $12x^5y^4z^3$.
11. $2x^4 + x^3 - 27x^2 + 41x - 14$.
13. $6x^4 - 25x^3 + 17x^2 + 28x - 20$.
15. $-x^4 - x^3 + 12x^2 - 7x - 5$.
17. $3x^2 - x + 4$.
19. $2x^2 - 7x - 4$.
21. $4, 3, 9, 18, 624, 4, 8a^2 - 6a + 4, 18a^2 - 9a + 4, 2a^2 + 5a + 6$.

Exercises II. A, page 23

1. -7.
3. 6.
5. $\frac{2}{5}$.
7. $\frac{d - b}{a - c}$.
9. No solution.
11. (a) $\frac{3y + 6}{4}$, (b) $\frac{4x - 6}{3}$.
13. (a) $\frac{3k - 7}{5k}$, (b) $\frac{7}{3 - 5x}$.
15. $\frac{Rd^2}{K}$.
17. 27, 24.
19. 47, 49, 51, 53, 55.
21. $3\frac{1}{2}$ in. $\times$ $6\frac{1}{2}$ in.
23. 35, 7.
25. 1230 at \$1.00, 770 at \$1.50.
27. 5 gal. regular, 7 gal. ethyl.
29. 40.
31. \$5650 at 3%, \$2350 at $3\frac{1}{2}$%.
33. 100 mi.
35. 450 mi.
37. 1528 ft. (approx.).
39. $43\frac{7}{11}$ min.
41. 9:20.
45. $5\frac{1}{4}$ ft. from 60-lb. child.
47. 150 lb.
49. 6 ft.

Exercises II. B, page 31

1. $28\frac{1}{2}, -3\frac{1}{2}$.
3. $7, -3$.
5. $-8, -11$.

7. 10, 4.
9. Inconsistent.
11. $-12, 17$.
13. $\frac{1}{17}, -\frac{21}{17}$.
15. 15, 2.
17. $7a, 4a$.
19. $4, -6$.
21. $\frac{3}{a}, \frac{5}{b}$.
23. $\frac{126a - 40b}{43a}, \quad \frac{105a + 24b}{43a}$
25(21). $\frac{3}{x}, \frac{5}{y}$.
27. $-1, 3, -5$.
29. $-3, 2, 7$.
31. $3, -2, 2$.
33. 2, 4, 6.
35. $\frac{2}{3}, -\frac{1}{2}, \frac{1}{4}$.
37. 2, 1, 0, 3.
39. $\frac{1}{2}, 1, \frac{3}{2}, 2$.

Exercises II. C, page 34

1. 57, 16.
3. 12, 18.
5. 24($1.50), 28($2.00).
7. $1250 at 3%, $6750 at 2½%.
9. 220 mi./hr., 20 mi./hr.
11. 840 mi., 100 mi.
13. 117 ft./sec., 39 ft./sec.
15. 32 lb. (15c.), 18 lb. (20c.).
17. 51($25), 18($50), 3($100).
19. 10 hr. (A), 15 hr. (B), 12 hr. (C).
21. 72.

Exercises III. A, page 45

1. $2ay(2x - 3z)$.
3. $(4x + 5y)(4x - 5y)$.
5. $2(3x^2 + 2y)(3x^2 - 2y)$.
7. $(3a - 4b)^2$.
9. $2(7a - 4x)^2$.
11. $(x - 6)(x + 2)$.
13. $(5 - x)(2 + x)$.
15. $(x - 2)(2x + 3)$.
17. $(3x + 2)(4x + 9)$.
19. $(3x - 4y)(6x - 5y)$.
21. $(a + 2)(3x + 5y)$.
23. $(a^2 + 1)(a - 1)$.
25. $(a + b)(a^2 - ab + b^2 + 1)$.
27. $(y + x - z)(y - x + z)$.
29. $(3a + 2b + x - y)(3a + 2b - x + y)$.
31. $(2a - b)^3$.
33. $(x^2 + 2)(x^4 - 2x^2 + 4)$.
35. $(a^2 + b^2)(a^4 - a^2b^2 + b^4)$.
37. $3(x^3 + 2y^3)(x^3 - 2y^3)$.
39. $(a^4 + b^4)(a^2 + b^2)(a + b)(a - b)$.
41. $(a^2 + b^4)(a^4 - a^2b^4 + b^8)$.
43. $2(2a^2 + 2ab + b^2)(2a^2 - 2ab + b^2)$.
45. $(3x^2 + 3xy - 2y^2)(3x^2 - 3xy - 2y^2)$.
47. $(5a^2 + 5ab + 7b^2)(5a^2 - 5ab + 7b^2)$.
49. $(x + y)(x - y)(x + 3y)(x - 3y)$.
51. $(2x^2 + 3y^2)(3x^2 - 5y^2)$.
53. $(x - 1)(x^2 + x - 3)$.
55. $a^b(a + c)(a - c)$.
57. 135.
59. 1997.

Exercises III. B, page 47

3. $x + y, (x + y)^2(x - y)$.
5. $3x + 2, (3x + 2)(2x - 5)(4x - 3)$.
7. $1, (81a^4 + 16b^4)(9a^2 + 4b^2)$.
9. $225ab^4c^2, 50625a^8b^6c^7$.

11. $x^2 - xy + y^2,\ x(x^3 + y^3)$. **13.** $a^2 - 2ab + 2b^2,\ a^2(a^4 + 4b^4)$.
15. $x^2 + y^2,\ (x^2 - y^2)(x^6 + y^6)$.

Exercises IV. A, page 51

1. $\dfrac{2yz^6}{3x^7}$. **3.** $\dfrac{x-4}{x+6}$. **5.** $\dfrac{3x-4}{x-2}$. **7.** $\dfrac{29}{36}$. **9.** $\dfrac{5x+2y}{x^2-4y^2}$.

11. $\dfrac{2x-16}{x^2-8x+15}$. **13.** $\dfrac{2b}{(a+b)^2(a-b)}$. **17.** $\dfrac{a}{3a-2b}$.

19. $\dfrac{x+y-2z}{(x-y)(y-z)(z-x)}$. **21.** $\dfrac{1+ab-b^2}{(a-b)^2}$.

23. $\dfrac{2a^2-5ab}{a-4b}$. **25.** $5x + 14 + \dfrac{68}{x-4}$.

27. $2x - \dfrac{4}{3} - \dfrac{5/3}{3x-2}$. **29.** $x + 4 - \dfrac{6x+6}{x^2+1}$.

Exercises IV. B, page 53

1. $\dfrac{9bxz^2}{10ac^4}$. **3.** $\dfrac{(x+1)(x+3)}{x+2}$. **5.** $\dfrac{21z^{12}}{16a^3b^{12}x^2y}$.

7. $(x^2+y^2)(x^2-xy+y^2)$. **9.** $\dfrac{a^4b^5c^6z}{x^3y}$.

11. $\dfrac{9r^5s^3tv}{10}$. **13.** $x(x+y)$. **15.** $\dfrac{y^2(x-y)}{x+y}$.

Exercises IV. C, page 55

1. $\dfrac{2}{9}$. **3.** $\dfrac{x+1}{x^2+1}$. **5.** $x + y$. **7.** $\dfrac{6}{x}$.

9. $\dfrac{x-1}{x}$. **11.** $\dfrac{5-2x}{2-x}$. **13.** $1 - x$.

15. $\dfrac{(x-1)(x-5)}{(x-3)^2}$.

Exercises IV. D, page 59

1. 13. **3.** -12. **5.** -3. **7.** No solution.
9. 3, -5. **11.** 2. **13.** 270 mi., 45 mi./hr.

Exercises V. A, page 63

1. 6. **3.** $3a^9$. **5.** $8a^6$. **7.** $2a^2$.

9. $\dfrac{5}{x}$. **11.** $\frac{1}{12}x^{7/12}$. **13.** $\frac{1}{3}x^{1/6}$. **15.** $\dfrac{b^2}{a^2}$.

17. $\dfrac{256}{x}$. **19.** $\frac{1}{8}$. **21.** $16a^8$. **23.** $\dfrac{9}{a^{18}}$.

25. $\dfrac{4a^8b^6c^{18}}{9x^2y^4}$. **27.** 1. **29.** $\dfrac{229}{1728}$. **31.** $\frac{9}{4}$.

33. $-\dfrac{(x-y)^2}{xy}$. **35.** $a^2(a+1)$.

37. $-\dfrac{a^2+b^2c^2}{2abc}$. **39.** $6x^3+7x^2-29x+12$.

41. $4x^{19/6}-13x^{13/6}-2x^{7/6}+15x^{1/6}$.

43. $2x^2+5x-3$. **45.** 4.56×10^6. **47.** 1.68×10^{-4}.

49. 3.5×10^6. **51.** 9.29×10^7. **53.** 7.594×10^{-6}.

55. 3.9×10^{13}. **57.** 5.1×10^{-4}. **59.** 2.5×10^{10}.

61. 5.87×10^{12}.

Exercises V. B, page 65

1. $7\sqrt{a}$. **3.** $3cde^2\sqrt[3]{2ab^2d^2e}$. **5.** $2uvw^3\sqrt[4]{4uv^3w^2}$.

7. $0.5x^{12}\sqrt{2x}$. **9.** $2x^{-6}\sqrt{3}$. **11.** $\dfrac{3bc\sqrt{3ac}}{2x^2y^3}$.

13. $2x\sqrt[5]{4x^2}$. **15.** $\sqrt{9xy}$. **17.** $\sqrt[3]{27a^4b^7}$.

19. $\sqrt{\dfrac{7}{x}}$. **21.** $\sqrt{x^2-y^2}$. **23.** $\sqrt[n]{a^{n+1}}$.

25. $\sqrt[n]{x^{mn+1}}$. **27.** $\sqrt{6a}$. **29.** $\sqrt{4a}$.

31. $\sqrt{\dfrac{5a}{13b}}$. **33.** $\sqrt[6]{xy^2z^3}$. **35.** $\sqrt{xy}$.

Exercises V. C, page 67

1. $\dfrac{2\sqrt{3}}{3}$. **3.** $\dfrac{3\sqrt[3]{4}}{2}$. **5.** $\dfrac{x\sqrt[3]{y^2}}{y}$.

7. $\dfrac{2\sqrt{a+b}}{a+b}$. **9.** $\dfrac{\sqrt[3]{12xy}}{2y}$. **11.** $\dfrac{\sqrt[5]{a^3}}{a}$.

13. $\dfrac{\sqrt{7}+\sqrt{2}}{5}$. **15.** $\dfrac{\sqrt{x}-\sqrt{y}}{x-y}$. **17.** $\dfrac{7-\sqrt{33}}{4}$.

19. $\dfrac{a+b-2\sqrt{ab}}{a-b}$. **21.** $\dfrac{2+\sqrt{6}-\sqrt{2}}{4}$.

23. $\dfrac{2\sqrt{3}+3\sqrt{2}+\sqrt{30}}{12}$. **27.** $\dfrac{a^{2/3}+a^{1/3}b^{1/3}+b^{2/3}}{a-b}$.

29. 2.121. **31.** −17.944. **33.** 5.940.

Exercises V. D, page 68

1. $3ab^2c^2\sqrt{2ac}$. **3.** $2xyz^5\sqrt[3]{2y^2z}$. **5.** $\dfrac{2a\sqrt{10ab}}{5b}$.

7. $\dfrac{\sqrt{2xy}}{2y}$. 9. $\dfrac{3\sqrt[3]{245ab^2}}{7b}$. 11. $\dfrac{\sqrt[3]{2x^2y}}{2xy}$.

13. $\dfrac{a^{1/2} - b^{1/2}}{a - b}$. 15. $\dfrac{\sqrt{11} + \sqrt{5}}{6}$. 17. $\dfrac{(7x^2)^{1/3}}{7x}$.

19. $3\sqrt{3} + \sqrt{6}$. 21. $5 - 2\sqrt{6}$. 23. $\dfrac{\sqrt{15} + \sqrt{10}}{5}$.

25. $xy\sqrt[n]{x^3y^5}$. 27. $b\sqrt{ab}$. 29. $\dfrac{1 + \sqrt{3} - \sqrt{2}}{2}$.

Exercises V. E, page 70

1. $9\sqrt{2}$. 3. $4\sqrt{2}$. 5. $\dfrac{5\sqrt{6x}}{6}$.

7. $14 + 2\sqrt{2} + 2\sqrt[5]{2}$. 9. $2^{1/2} - 260$.

11. $\dfrac{8\sqrt{15}}{9}$. 13. $\left(24x + \dfrac{1}{10} + \dfrac{10}{x^2}\right)\sqrt{10x}$.

15. $\sqrt{78}$. 17. $7x^2\sqrt[3]{x^2}$. 19. $\sqrt[6]{72}$.

21. $2\sqrt[6]{54}$. 23. $\dfrac{3^{1/3}}{3}$. 25. $\frac{1}{12}\sqrt[12]{a^7}$.

27. $81\sqrt{3}$. 29. $3 - \sqrt[3]{4}$.

31. $2\sqrt{3} - 2\sqrt[6]{72} + 3\sqrt[6]{864} - 6\sqrt[3]{6}$.

33. $x + \sqrt{x} - \sqrt[3]{x^2} + 2\sqrt[4]{x^3}$. 35. $x^2 - 5x + \frac{13}{4}$.

37. $\sqrt[3]{25}$. 39. $\dfrac{\sqrt[6]{3125}}{5}$.

41. $\dfrac{\sqrt{30}}{6}$. 43. $\dfrac{\sqrt[12]{32a^7b^2c^5}}{a}$.

45. $\dfrac{\sqrt[6]{x^5y^4z^3}}{yz}$. 47. $\sqrt[12]{a^2b^5}$. 49. 0. 51. 0.

53. $p^2 + p\sqrt{p^2 - 4q}$. 55. 0.

Exercises V. F, page 74

1. $6 + 3i$. 3. $7 - i\sqrt{6}$. 5. $\frac{9}{25} + \frac{3}{5}i$. 7. $-\frac{3}{4} + \frac{1}{2}i$.

9. $\dfrac{4}{3} + \dfrac{2\sqrt{3}}{3}i$. 11. $-2 + 2\sqrt{2}\cdot i$. 13. $6 + 8i$.

15. $1 - 6i$. 17. $15 - 3i$. 19. $8 + 5\sqrt{2}\cdot i$.

21. $\dfrac{13}{6} + \dfrac{5\sqrt{6}}{6}i$. 23. $\sqrt{15} + 30 + (5\sqrt{10} - 3\sqrt{6})i$.

25. $5 + 3i$. 27. $-13 + 6\sqrt{3}\cdot i$.

29. $2\sqrt{3} + 3\sqrt{2} + (3\sqrt{2} - 2\sqrt{3})i$.

31. $-9 + 14i$. 33. $\frac{14}{5} - \frac{2\sqrt{10}}{5}i$. 35. $2 + 23i$.

37. $-16 + 19\sqrt{5} \cdot i$. 39. $-12 - 27\sqrt{3} \cdot i$. 41. 194.

43. $-\frac{1}{6} - \frac{\sqrt{6}}{3}i$. 45. $a^2 + b^2$. 47. $-2 + 2i$.

49. 1. 51. $\frac{23}{13} - \frac{2}{13}i$. 53. $-\frac{32}{85} + \frac{9}{85}i$.

55. $-\frac{54}{49} - \frac{47\sqrt{6}}{49}i$. 57. $\frac{3}{52} + \frac{19\sqrt{3}}{52}i$. 59. $\frac{1}{7} + \frac{4\sqrt{3}}{7}i$.

61. $\frac{5\sqrt{6} - 3\sqrt{2}}{12} + \frac{\sqrt{3}}{6}i$. 63. $\frac{a}{a^2 + b^2} + \frac{b}{a^2 + b^2}i$.

65. $\frac{a^2 - b^2}{(a^2 + b^2)^2} + \frac{2ab}{(a^2 + b^2)^2}i$. 67. 0.

Exercises VI. A, page 81

1. 2, 7. 3. $-2, -6$. 5. 1, 5. 7. 4, 4. 9. $1 \pm 2i$.

11. 4, 6. 13. 1, 125. 15. ± 7. 17. $\pm 7i$. 19. $\pm 2\sqrt{2} \cdot i$.

21. 4, 24. 23. $6 \pm \sqrt{14}$. 25. $\frac{-5 \pm i\sqrt{3}}{2}$.

27. $\frac{1}{3}, -\frac{3}{2}$. 29. $-\frac{2}{3}, -\frac{3}{4}$. 31. $\frac{11 \pm i\sqrt{55}}{8}$.

33. $\frac{2 \pm \sqrt{3}}{5}$. 35. $\frac{1 \pm 2\sqrt{6} \cdot i}{5}$. 37. $\frac{5}{3}, -\frac{2}{7}$.

39. $\frac{5 \pm \sqrt{10}}{5}$. 41. $\frac{8 \pm \sqrt{115}}{17}$. 43. $\frac{\sqrt{5} \pm 7}{22}$.

45. $\frac{5\sqrt{7} \pm \sqrt{155}}{5}$. 47. $\frac{7}{3}, -\frac{2}{5}$. 49. 0, 5.

51. $4, \frac{11}{2}$. 53. 0.4, 0.5. 55. $3 \pm \sqrt{k}$.

57. $\frac{3(1 \pm \sqrt{k})}{1 - k}$. 59. $6y, 2y$. 61. $(4 \pm \sqrt{2})y$.

63. $(-2 \pm i\sqrt{6})y$. 65. $\frac{5y}{2}, -\frac{3y}{8}$. 67. $1 \pm \sqrt{1 - 4y - y^2}$.

69. $4 - 2y \pm \sqrt{2(5 - 2y - y^2)}$. 70(59). $\frac{x}{2}, \frac{x}{6}$.

70(61). $\frac{4 \pm \sqrt{2}}{14}x$. 70(63). $\frac{-2 \pm i\sqrt{6}}{10}x$.

70(65). $\frac{2x}{5}, -\frac{8x}{3}$.

70(67). $-2 \pm \sqrt{4 + 2x - x^2}$.

70(69). $1 - \frac{x}{3} \pm \frac{1}{6}\sqrt{2x(12 - x)}$.

71. 17 in. × 22 in.

73. 3 ft. **75.** 23, 24. **77.** 32, 34. **79.** 60.

81. 20 in. × 24 in. **83.** 10 ft. **85.** 2 mi./hr.

87. 45 mi./hr. **89.** 24 min.

93. $(x + 1)^2 + y^2 = 49$, $(-1,0)$, 7.

95. $(x - 3)^2 + (y + 1)^2 = 16$, $(3,-1)$, 4.

97. $(x - 3)^2 + (y - 4)^2 = 25$, $(3,4)$, 5.

99. $(x + 5)^2 + (y - 1)^2 = 1$, $(-5,1)$, 1.

101. $(x + 11)^2 + (y + 15)^2 = 576$, $(-11,-15)$, 24.

103. $\frac{(x+3)^2}{16} + \frac{(y-2)^2}{4} = 1$, $(-3,2)$, $a = 4$, $b = 2$.

105. $\frac{(x+2)^2}{25} + \frac{(y+1)^2}{16} = 1$, $(-2,-1)$, $a = 5$, $b = 4$.

107. $\frac{(x+5)^2}{75} + \frac{(y-3)^2}{27} = 1$, $(-5,3)$, $a = 5\sqrt{3}$, $b = 3\sqrt{3}$.

109. $\frac{(x-1)^2}{27} + \frac{(y+2)^2}{5} = 1$, $(1,-2)$, $a = 3\sqrt{3}$, $b = \sqrt{5}$.

111. $\frac{(x+6)^2}{9/4} + \frac{(y+2)^2}{4/9} = 1$, $(-6,-2)$, $a = \frac{3}{2}$, $b = \frac{2}{3}$.

115. $\frac{(x-3)^2}{16} - \frac{(y+4)^2}{25} = 1$, $(3,-4)$, $a = 4$, $b = 5$.

117. $\frac{(x+2)^2}{144} - \frac{(y-2)^2}{9} = 1$, $(-2,2)$, $a = 12$, $b = 3$.

119. $\frac{(x-7)^2}{4} - \frac{(y-4)^2}{4} = 1$, $(7,4)$, $a = 2$, $b = 2$.

121. $\frac{(x+1)^2}{8} - \frac{(y-2)^2}{6} = 1$, $(-1,2)$, $a = 2\sqrt{2}$, $b = \sqrt{6}$.

123. $\frac{(x+\frac{1}{2})^2}{6} - \frac{(y+3)^2}{4/3} = 1$, $(-\frac{1}{2},-3)$, $a = \sqrt{6}$, $b = \frac{2\sqrt{3}}{3}$.

125. $\sqrt{5}\sqrt{(x-4)^2 - 4}$.

127. $2\sqrt{(x+5)^2 - \frac{9}{4}}$.

129. $\sqrt{2}\sqrt{(x+\frac{1}{2})^2 + \frac{1}{4}}$.

131. $\sqrt{3}\sqrt{(x-\frac{1}{6})^2 - \frac{25}{36}}$.

133. $\sqrt{3}\sqrt{(x+\frac{2}{3})^2 - \frac{4}{9}}$.

135. $\sqrt{6}\sqrt{6 - (x-1)^2}$.

137. $\sqrt{2}\sqrt{\frac{7}{4} - (x+\frac{1}{2})^2}$.

139. $\sqrt{9 - (x-3)^2}$.

141. $\sqrt{2}\sqrt{\frac{17}{16} - (x+\frac{1}{4})^2}$.

143. $2\sqrt{\frac{21}{16} - (x-\frac{3}{4})^2}$.

Exercises VI. B, page 88

1. $\pm 2, \pm 3$.

3. $\pm\frac{4}{3}, \pm\frac{\sqrt{6}}{2}i$.

5. $1, -2, 1 \pm i\sqrt{3}, \frac{-1 \pm i\sqrt{3}}{2}$.

7. $\frac{1}{4}, \frac{4}{25}$. 9. 8, 27. 11. 1296. 13. $1, \frac{1}{64}$.
15. $729, \frac{1}{64}$. 17. $\frac{1}{2}, \frac{1}{5}$. 19. $1, -2, 2 \pm \sqrt{6}$.
21. $-1, -\frac{13}{8}$. 23. $\frac{3 \pm \sqrt{17}}{2}, \frac{3 \pm 3\sqrt{3} \cdot i}{2}$.

25. 3, −1. 27. $0, -5, \frac{-5 \pm \sqrt{17}}{2}$.

29. 497 ft. approx.

Exercises VI. C, page 92

1. 41. 3. $\frac{3}{2}, \frac{5}{8}$. 5. $-\frac{1}{2}$. 7. 10, 22.
9. 25. 11. No solution. 13. 2. 15. $\frac{6}{5}$.
17. 3. 19. 1. 21. 25 ft., 26 ft. 23. 13 mi.

Exercises VI. D, page 95

1. Real, unequal, irrational. 3. Imaginary.
5. Real, unequal, irrational. 7. Real, equal, rational.
9. Real, unequal, rational, numerically equal.
11. Imaginary. 13. Real, unequal, irrational.
15. Imaginary. 19. $\frac{7}{8}$. 21. $-\frac{21}{4}$. 23. $0, \frac{16}{7}$.
25. 3. 27. Any value. 29. −1, −3. 31. Impossible.
33. 2. 35. $\pm 2\sqrt{3}$. 37. Impossible. 39. 1, 4.
41. ±3. 43. Real, unequal, irrational.
45. Real, equal, irrational. 47. Imaginary.

Exercises VI. E, page 100

1. −5, 3. 3. $\frac{1}{2}, -\frac{1}{4}$. 5. $-\frac{3}{2}, -\frac{4}{3}$. 7. $-\frac{4}{3}, 0$. 9. $\frac{6}{25}, 0$.
11. $k + 1, \frac{2k}{2 - k}$ $(k \neq 2)$. 13. $-\frac{21}{4}$. 15. $\frac{23}{3}$. 17. 12.

19. 4. 21. a.
23. $x^2 - 12x + 35 = 0$. 25. $x^2 - 8x + 16 = 0$.
27. $32x^2 + 4x - 15 = 0$. 29. $x^2 - 3 = 0$.
31. $x^2 + 5 = 0$. 33. $x^2 + 6x + 2 = 0$.
35. $x^2 - 4x + 7 = 0$. 37. $9x^2 - 30x + 28 = 0$.
39. $225x^2 - 150x + 43 = 0$. 41. $x^2 - 2\sqrt{2} \cdot x - 1 = 0$.
43. $x^2 - 2\sqrt{2} \cdot x + 5 = 0$. 45. $18x^3 - 33x^2 - 58x - 15 = 0$.

Exercises VI. F, page 101

1. $(12x + 5)(2x - 125)$. 3. $(2x + 1)(32x - 49)$.

7. $(x + i\sqrt{5})(x - i\sqrt{5})$. 9. $(x + 3 + 2i)(x + 3 - 2i)$.

11. $(3x - 2 - \sqrt{7})\left(x - \dfrac{2 - \sqrt{7}}{3}\right)$.

13. $(x - \sqrt{2} + \sqrt{3})(x - \sqrt{2} - \sqrt{3})$.

15. $(x - \sqrt{5} - i\sqrt{3})(x - \sqrt{5} + i\sqrt{3})$.

Exercises VI. G, page 104

13. 0, min.; 2. 15. $-9\frac{1}{4}$, min.; $-2\frac{1}{2}$. 17. 9, max.; -1.
19. $-7\frac{1}{2}$, min.; $1\frac{1}{2}$. 21. 4, max.; 2. 23. $\frac{3}{4}$, min.; $\frac{1}{2}$.
25. $-2\frac{1}{2}$, min.; -1. 27. $\frac{1}{9}$, min.; $\frac{1}{3}$. 29. $\frac{1}{2}$, min.; $-\dfrac{\sqrt{2}}{2}$.
31. 9, 9. 35. 80¢. 37. 7500 sq. ft.

Exercises VII. A, page 107

1. (3,4), (4,−3). 3. (0,0), $(-1, \frac{1}{3})$.
5. $(2 + i\sqrt{2}, 2 - i\sqrt{2})$, $(2 - i\sqrt{2}, 2 + i\sqrt{2})$.
7. (6,2), (−3,−4). 9. (8,−12), (−9,5).
11. $(2 + \sqrt{3}, 3 + 2\sqrt{3})$, $(2 - \sqrt{3}, 3 - 2\sqrt{3})$.
13. $\left(-3 + \sqrt{11}, \dfrac{10 - 3\sqrt{11}}{2}\right)$, $\left(-3 - \sqrt{11}, \dfrac{10 + 3\sqrt{11}}{2}\right)$.
15. (4,3), $(-\frac{13}{5}, -\frac{7}{5})$. 17. (1,2), (4,−2). 19. (1,−1), (−1,2).
21. No solution. 23. 11, 15. 25. 9 ft. × 40 ft.

Exercises VII. B, page 109

1. (4,3), (4,−3), (−4,3), (−4,−3).
3. $(3,2i)$, $(3,-2i)$, $(-3,2i)$, $(-3,-2i)$.
5. $(2\sqrt{2}, \sqrt{3})$, $(2\sqrt{2}, -\sqrt{3})$, $(-2\sqrt{2}, \sqrt{3})$, $(-2\sqrt{2}, -\sqrt{3})$.
7. (5,0), (−5,0), each double.
9. $(2\sqrt{3} \cdot i, i\sqrt{5})$, $(2\sqrt{3} \cdot i, -i\sqrt{5})$, $(-2\sqrt{3} \cdot i, i\sqrt{5})$, $(-2\sqrt{3} \cdot i, -i\sqrt{5})$.
11. $\left(\dfrac{6\sqrt{5}}{5}, \dfrac{4\sqrt{5}}{5}\right)$, $\left(\dfrac{6\sqrt{5}}{5}, -\dfrac{4\sqrt{5}}{5}\right)$, $\left(-\dfrac{6\sqrt{5}}{5}, \dfrac{4\sqrt{5}}{5}\right)$, $\left(-\dfrac{6\sqrt{5}}{5}, -\dfrac{4\sqrt{5}}{5}\right)$. 13. $4\sqrt{35}$, $2\sqrt{65}$.

Exercises VII. C, page 112

1. (4,3), (4,−3), (−4,3), (−4,−3).
3. (4,4), (−4,−4).
5. (6,−3), (−6,3), $(5\sqrt{3}, 4\sqrt{3})$, $(-5\sqrt{3}, -4\sqrt{3})$.

7. $(\sqrt{3}, 2\sqrt{3}), (-\sqrt{3}, -2\sqrt{3}), \left(\frac{11\sqrt{111}}{37}, \frac{\sqrt{111}}{37}\right),$
$\left(-\frac{11\sqrt{111}}{37}, -\frac{\sqrt{111}}{37}\right).$

9. $(1,2), (-1,-2), \left(\frac{4\sqrt{114}}{19}i, \frac{\sqrt{114}}{57}i\right), \left(-\frac{4\sqrt{114}}{19}i, -\frac{\sqrt{114}}{57}i\right).$

11. $(2,-1), (-2,1), \left(\frac{4\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}\right), \left(-\frac{4\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right).$

13. $(3,-3), (-3,3), \left(\frac{12\sqrt{58}}{29}, -\frac{21\sqrt{58}}{58}\right), \left(-\frac{12\sqrt{58}}{29}, \frac{21\sqrt{58}}{58}\right).$

15. 7 in. $\times$ 24 in.

Exercises VII. D, page 115

1. $(3,-2), (-2,3), (2,-3), (-3,2).$
3. $(5,4), (4,5), (-7 + 3\sqrt{7}\cdot i, -7 - 3\sqrt{7}\cdot i),$
$(-7 - 3\sqrt{7}\cdot i, -7 + 3\sqrt{7}\cdot i).$
5. $(3,-2), (-2,3), (\frac{23}{11}, -\frac{37}{11}), (-\frac{37}{11}, \frac{23}{11}).$
7. $(1 + \sqrt{2}, 1 - \sqrt{2}), (1 - \sqrt{2}, 1 + \sqrt{2}),$
$\left(\frac{-1 + \sqrt{37}}{4}, \frac{-1 - \sqrt{37}}{4}\right), \left(\frac{-1 - \sqrt{37}}{4}, \frac{-1 + \sqrt{37}}{4}\right).$
9. $(6,1), (1,6), \left(\frac{-9 + i\sqrt{23}}{2}, \frac{-9 - i\sqrt{23}}{2}\right),$
$\left(\frac{-9 - i\sqrt{23}}{2}, \frac{-9 + i\sqrt{23}}{2}\right).$
11. 20 in., 21 in.

Exercises VII. E, page 118

1. $(7,5), (-7,-5), (5,7), (-5,7).$
3. $(5,5), (-5,-5).$
5. $(3,-4), (-3,4), (18, -\frac{13}{2}), (-18, \frac{13}{2}).$
7. $(5,3), (-3,-5).$
9. $(3,1), (3,-1), (-3,1), (-3,-1).$
11. $(3, \frac{1}{3}), (-\frac{1}{10}, -10), (2,-3), (\frac{9}{10}, -\frac{20}{3}).$
13. $(1,-2), (-2,4), \left(\frac{3 + \sqrt{137}}{8}, \frac{9 + 3\sqrt{137}}{16}\right),$
$\left(\frac{3 - \sqrt{137}}{8}, \frac{9 - 3\sqrt{137}}{16}\right).$
15. $(4i,3i), (-4i,-3i), \left(\frac{3\sqrt{10}}{5}, -\frac{3\sqrt{10}}{5}\right), \left(-\frac{3\sqrt{10}}{5}, \frac{3\sqrt{10}}{5}\right).$
17. $(1,-3), (-1,3), (4,-2), (-4,2).$

21. $3 + \sqrt{2}$.
23. $\sqrt{7} - \sqrt{2}$.
25. $\sqrt{15} + \sqrt{6}$.
27. $3\sqrt{15} + 5\sqrt{10}$.

Exercises VII. F, page 121

3. ± 2.
5. 13.
7. $-\frac{5}{6}$.
9. ± 4.
11. -5.
13. ± 16.
15. $\pm r\sqrt{1 + m^2}$.
17. $\pm\sqrt{a^2m^2 - b^2}$.

Exercises VII. G, page 123

1. $(3,3), (-2,-\frac{9}{2})$.
3. $(2,6), (-2,6), (\frac{3}{2}i,-\frac{13}{2}), (-\frac{3}{2}i,-\frac{13}{2})$.
5. $(4,7), (-3,0)$.
7. $(3,-2), (-3,2), \left(\frac{13\sqrt{105}}{35}, \frac{2\sqrt{105}}{35}\right), \left(-\frac{13\sqrt{105}}{35}, -\frac{2\sqrt{105}}{35}\right)$.
9. $(3,-2), (-2,3), \left(\frac{8 + 2\sqrt{7}\cdot i}{5}, \frac{8 - 2\sqrt{7}\cdot i}{5}\right),$
$\left(\frac{8 - 2\sqrt{7}\cdot i}{5}, \frac{8 + 2\sqrt{7}\cdot i}{5}\right)$.
11. $(4i,-2i), (-2i,4i), (-4i,2i), (2i,-4i)$.
13. $(3\sqrt{6} + 2\sqrt{5}, 3\sqrt{6} - 2\sqrt{5}), (3\sqrt{6} - 2\sqrt{5}, 3\sqrt{6} + 2\sqrt{5}),$
$(-3\sqrt{6} + 2\sqrt{5}, -3\sqrt{6} - 2\sqrt{5}),$
$(-3\sqrt{6} - 2\sqrt{5}, -3\sqrt{6} + 2\sqrt{5})$.
15. $(4,5), (-\frac{5}{2},-8)$.
17. $(6,\frac{1}{2}), (-16,-\frac{3}{16}), (-2,\frac{1}{4}), (-8,\frac{1}{16})$.
19. $(1,\frac{1}{2}), (-1,-\frac{1}{2}), \left(\frac{\sqrt{7}}{5}, \sqrt{7}\right), \left(-\frac{\sqrt{7}}{5}, -\sqrt{7}\right)$.
21. $(5,2), (-\frac{45}{19},-\frac{18}{19}), \left(\frac{-8 + 2\sqrt{326}\cdot i}{19}, \frac{12 - 3\sqrt{326}\cdot i}{19}\right),$
$\left(\frac{-8 - 2\sqrt{326}\cdot i}{19}, \frac{12 + 3\sqrt{326}\cdot i}{19}\right)$.
23. $(5,3), (-3,-5), \left(\frac{-5 + 5\sqrt{3}\cdot i}{2}, \frac{-3 + 3\sqrt{3}\cdot i}{2}\right),$
$\left(\frac{-5 - 5\sqrt{3}\cdot i}{2}, \frac{-3 - 3\sqrt{3}\cdot i}{2}\right), \left(\frac{3 + 3\sqrt{3}\cdot i}{2}, \frac{5 + 5\sqrt{3}\cdot i}{2}\right),$
$\left(\frac{3 - 3\sqrt{3}\cdot i}{2}, \frac{5 - 5\sqrt{3}\cdot i}{2}\right)$.
25. $(5,-3), (-3,5), (1 + i\sqrt{22}, 1 - i\sqrt{22}), (1 - i\sqrt{22}, 1 + i\sqrt{22})$.
27. $(2,\sqrt{3},3i), (2,\sqrt{3},-3i), (2,-\sqrt{3},3i), (2,-\sqrt{3},-3i), (-2,\sqrt{3},3i),$
$(-2,\sqrt{3},-3i), (-2,-\sqrt{3},3i), (-2,-\sqrt{3},-3i)$.
29. $(2,-2,1), (-4,0,-3)$.
31. $(8,-4,2), (2,-4,8)$.
33. 7 in., 24 in.
35. 12, 24.

37. 8 in., 15 in., 17 in. **39.** 108.
41. 4 hr. 40 min., 3 hr. 30 min. **43.** 27.
45. $\frac{2\sqrt{46}}{3}, \frac{2\sqrt{79}}{3}, \frac{2\sqrt{106}}{3}$. **47.** 7 ft., 9 lb.
49. $5\frac{1}{2}$ in. $\times$ $8\frac{1}{2}$ in., $7\frac{2}{7}$ in. $\times$ $6\frac{5}{12}$ in.
51. 7 mi./hr., 2 mi./hr. **53.** 1 hr. 45 min., 2 hr. 20 min.

Exercises VIII. B, page 133

1. $x > \frac{7}{3}$. **3.** $x > \frac{3}{2}$. **5.** $x > \frac{26}{9}$.
7. $x < -1$ or $x > 6$. **9.** $x < 2 - \sqrt{2}$ or $x > 2 + 2\sqrt{2}$.
11. $x < -3$ or $x > 5$. **13.** $x < -4$ or $x > \frac{3}{2}$.
15. x = any real number. **17.** $x < \frac{3}{2}$ or $x > \frac{9}{4}$.
19. $-2 < x < 1$ or $x > 3$. **21.** $x > 1$ but $\neq 3$.
23. $-1 < x < 0$ or $x > 1$. **25.** $x < -3$ or $x > 4$.
27. $x > -3$ but $\neq 1$ or 4. **29.** $-13 < x < -3$ or $x > 2$.
31. $-\frac{3}{2} < x < 1$ or $x > 3$. **33.** $-7 < x < \frac{1}{2}$ or $x > 3$.
35. $k < -4$ or $k > 0$. **37.** $\frac{-1 - \sqrt{2}}{2} < k < \frac{-1 + \sqrt{2}}{2}$.
39. $k \neq \frac{1}{2}$. **41.** $k < -\frac{1}{3}$. **43.** $k < 3 - 2\sqrt{3}$ or $k > 3 + 2\sqrt{3}$.
45. $-20 < k < 20$. **47.** $k < 1$. **49.** $-25 < k < 25$.
51. $k < -\frac{24}{7}$ or $k > \frac{24}{7}$. **53.** $-5 < k < 5$.

Exercises IX. A, page 137

3. 16, 9. **5.** 3, $\frac{27}{4}$. **7.** $\frac{48}{5}, \frac{1}{16}$. **9.** 6. **11.** $\frac{3}{2}$.
13. 940 r.p.m.
15. (a) 1.28 lb., (b) 3.125 lb., (c) 40 mi./hr.
17. 1 day. **19.** 16 ft.
21. (a) 22.5 lb./sq. in., (b) 18 lb./sq. in.
23. 2560 lb. **25.** 6.4 lb. **27.** $26\frac{2}{3}$.
29. 14.7 amp. **31.** 11.86 yr. **33.** $\frac{d\sqrt{6}}{4}, \frac{d\sqrt{15}}{10}$.

Exercises X. A, page 146

25. False. **27.** True. **29.** True. **31.** True. **33.** True.
35. False. (True for $n \leqq 40$ but fails for $n = 41$.)

Exercises X. B, page 148

1. $x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$.
3. $1 + 6x + 15x^2 + 20x^3 + 15x^4 + 6x^5 + x^6$.
5. $a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7$.
7. $a^8 - 8a^7b + 28a^6b^2 - 56a^5b^3 + 70a^4b^4 - 56a^3b^5 + 28a^2b^6 -$
$8ab^7 + b^8$.

9. $x^{10} + 10x^9y + 45x^8y^2 + 120x^7y^3 + 210x^6y^4 + 252x^5y^5 + 210x^4y^6 + 120x^3y^7 + 45x^2y^8 + 10xy^9 + y^{10}$.
11. $8x^3 + 36x^2y + 54xy^2 + 27y^3$.
13. $16x^8 + 32x^6\sqrt{y} + 24x^4y + 8x^2y\sqrt{y} + y^2$.
15. $\frac{1}{64}x^{18} - \frac{9}{16}x^{15}y^2 + \frac{135}{16}x^{12}y^4 - \frac{135}{2}x^9y^6 + \frac{1215}{4}x^6y^8 - 729x^3y^{10} + 729y^{12}$.
17. $x^6y^{12} - 18x^5y^{10}z^3 + 135x^4y^8z^6 - 540x^3y^6z^9 + 1215x^2y^4z^{12} - 1458xy^2z^{15} + 729z^{18}$.
19. $0.00032x^5 + 0.00640x^4y + 0.05120x^3y^2 + 0.20480x^2y^3 + 0.40960xy^4 + 0.32768y^5$.
21. $x^{25} + 25x^{24}y + 300x^{23}y^2 + 2300x^{22}y^3$.
23. $x^3 + 3x^2y + 3xy^2 + y^3 - 3x^2z - 6xyz - 3y^2z + 3xz^2 + 3yz^2 - z^3$.
25. 104.8576. **27.** 112,550,881. **29.** 1,159,274,074,3.
31. 997,002,999. **33.** 1,061,520,150,601.
35. 1.1610. **37.** 1.4859.

Exercises X. C, page 151

3. $2{,}449{,}440x^{12}y^{12}$. **5.** $-\frac{309375}{16}a^{5/2}b^{35}$. **7.** $1{,}451{,}520a^{16}b^{12}$.
9. $-3003a^{10}b^5$. **11.** $15120x^9y^8$. **13.** $1260x^2y^6z^2$.
15. 8.

Exercises X. D, page 155

1. $1 - x + x^2 - x^3$. **3.** $1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3$.
5. $1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3$. **7.** $1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}x^3$.
9. $1 + \frac{1}{5}x - \frac{2}{25}x^2 + \frac{6}{125}x^3$. **11.** $1 + 0.4x - 0.12x^2 + 0.064x^3$.

The first number in the answers to exercises 13–35 is obtained by using three terms of the binomial series. The number in parentheses is the correct approximation, accurate to the number of places given.

13. 1.0392 (1.0392). **15.** 0.994,987,5 (0.994,987,4).
17. 9.8995 (9.8995). **19.** 4.8990 (4.8990).
21. 7.071064 (7.071068). **23.** 5.657,407,41 (5.656,856).
25. 1.0196 (1.0196). **27.** 10.00999 (10.00999).
29. 2.153 (2.154). **31.** 3.03658 (3.03659).
33. 5.03968 (5.03968). **35.** 0.9898 (0.9898).

Exercises XI. A, page 159

1. 48, 255. **3.** 218, 2750. **5.** −22, −66.
7. 59.9, 3020.0. **9.** $2\frac{1}{6}$, $29\frac{1}{6}$. **11.** 13, 234.
13. $1\frac{1}{4}$, 108.
15. $14\frac{1}{3}$, $15\frac{2}{3}$, 17, $18\frac{1}{3}$, $19\frac{2}{3}$, 21, $22\frac{1}{3}$, $23\frac{2}{3}$. **17.** $138\frac{1}{3}$, $12687\frac{1}{2}$.
19. $-3\frac{1}{3}$, 10. **21.** $-11\frac{1}{2}$, $30\frac{1}{2}$. **23.** 9, 59. **25.** 7, 14.
27. 13, $2\frac{2}{3}$. **29.** 74. **31.** 31. **33.** 0, 4.

35. $n(n+1)$.

37. $a_1 = a_n - (n-1)d$, $S_n = \frac{n}{2}[2a_n - (n-1)d]$.

39. $a_1 = \frac{2S_n}{n} - a_n$, $d = \frac{2(na_n - S_n)}{n(n-1)}$.

41. $n = \frac{2S_n}{a_1 + a_n}$, $d = \frac{a_n^2 - a_1^2}{2S_n - a_1 - a_n}$. **43.** $d = \frac{a_n - a_1}{n-1}$.

Exercises XI. B, page 163

1. 972, 1456. **3.** −98304, −78642. **5.** 159,744, 319,332.
7. $\frac{54}{625}$, $\frac{2882}{1875}$. **9.** 1.125,508,81, 5.309,135,81.
11. $1458\sqrt{3}$, $2186(1+\sqrt{3})$. **13.** $\frac{\sqrt{2}}{32}$, $\frac{63\sqrt{2}}{32}$.
15. 5, $105\frac{1}{2}$. **17.** $\frac{3}{2}$, 1055; $-\frac{3}{2}$, 275.
19. ±35, 245, ±1715. **21.** 16, 1031.
23. 7, 1701. **25.** 5, 0.064. **27.** 9, 3066. **29.** $\frac{4}{3}$, 5.
31. 52, 5. **33.** 6, 252; −7, 343. **35.** $\pm 4\sqrt{3}$.
37. 25, −1.

39. $a_1 = \frac{r-1}{r^n - 1} S_n$, $a_n = \frac{r-1}{r^n - 1} r^{n-1} S_n$.

41. $a_n = \frac{a_1 + (r-1)S_n}{r}$. **43.** $r = \frac{S_n - a_1}{S_n - a_n}$.

Exercises XI. C, page 165

1. $\frac{5}{2}$. **3.** $\frac{21}{5}$. **5.** $-\frac{384}{5}$.
7. $\frac{3(3+\sqrt{3})}{2}$. **9.** $\frac{32(5+4\sqrt{5})}{11}$. **11.** $1+\sqrt{5}$.
13. $\frac{4}{9}$. **15.** $\frac{4}{11}$. **17.** 51.

Exercises XI. D, page 167

1. $\frac{2}{3}$. **3.** $\frac{7}{9}$. **5.** $\frac{32}{99}$.
7. $2\frac{34}{99}$. **9.** $\frac{241}{990}$. **11.** $\frac{61}{495}$.
13. $\frac{1}{7}$. **15.** $\frac{1234}{9999}$. **17.** $\frac{10}{11}$.

Exercises XI. E, page 168

1. $\frac{1}{59}$. **3.** $\frac{3}{25}$. **5.** $\frac{2}{5}$, $\frac{1}{3}$, $\frac{2}{7}$.
7. 12, 15, 20, 30. **9.** 3. **11.** 5, −4.

Exercises XI. F, page 169

1. 88, 520. **3.** $\frac{112}{2187}$, $\frac{59017}{2187}$. **5.** −230.5, −10675.
7. 192, 129. **9.** $\frac{6}{5}$.

11. (a) 45 ft., (b) 192 ft., (c) 10 sec. **13.** 6th.
15. (a) $113\frac{2}{3}$ ft., (b) 135 ft. **17.** 352800 ft.-lb.
19. $4\frac{191}{256}$ gal. **21.** 32.768 in.
23. 226.
25. (a) 72π sq. in., (b) 144 sq. in.
27. (a) $6(1 + \sqrt{2})$ in., (b) $6\sqrt{3}$ in., (c) $6(2 + \sqrt{3})$ in.
29. 1, 5, 9, 13.

Exercises XII. A, page 176

1. $10 + 9i$. **3.** $-3 + 3i$. **5.** $-2 - 2i$.
7. $-8 + 7i$. **9.** $-i$. **11.** $8 - 5\sqrt{2} \cdot i$.
13. $3 - 2\sqrt{2} \cdot i$. **15.** $1 + 47i$. **17.** $22 + 63i$.
19. $65 + 74i$. **21.** $\frac{23}{24} + \frac{13}{72}i$. **23.** $-156 + 158i$.
25. $43 - 11\sqrt{2} \cdot i$. **27.** $4 + 17\sqrt{2} \cdot i$. **29.** $11 - 60i$.
31. $\frac{41}{65} + \frac{23}{65}i$. **33.** $\frac{58}{73} + \frac{33}{73}i$. **35.** $-\frac{95}{109} - \frac{26}{109}i$.
37. $-\dfrac{21}{68} - \dfrac{67}{68}i$. **39.** $\dfrac{13}{123} + \dfrac{31\sqrt{2}}{123}i$. **41.** $\dfrac{8}{11} - \dfrac{13\sqrt{2}}{22}i$.
43. $\frac{11}{61} + \frac{60}{61}i$. **45.** $\frac{147}{13} - \frac{71}{13}i$. **47.** $\frac{513}{754} - \frac{269}{754}i$.

Exercises XII. C, page 180

For abbreviation, cis A is written for $\cos A + i \sin A$.

3. $3\sqrt{2}$ cis 45°. **5.** 5 cis 323.1°. **7.** 5 cis 90°.
9. 10 cis 240°. **11.** 13 cis 112.6°. **13.** 9 cis 270°.
15. 10 cis 143.1°. **17.** 29 cis 46.4°. **19.** 3 cis 118.8°.
21. $3\sqrt{3} + 3i$. **23.** $2\sqrt{2} + 2\sqrt{2} \cdot i$. **25.** -3.
27. $-\sqrt{3} - i$. **29.** $-1 + i$. **31.** $\dfrac{\sqrt{3}}{2} - \dfrac{1}{2}i$.
33. $-19.318 + 5.176i$. **35.** $7.970 + 0.697i$.

Exercises XII. D, page 182

1. 6 cis 70° = $2.052 + 5.638i$. **3.** 14 cis 300° = $7 - 7\sqrt{3} \cdot i$.
5. $20\sqrt{2}$ cis 165° = $-10(\sqrt{3} + 1) + 10(\sqrt{3} - 1)i$.
7. 2 cis 230° = $-1.286 - 1.532i$.
9. $3\sqrt{2}$ cis 210° = $-\dfrac{3\sqrt{6}}{2} - \dfrac{3\sqrt{2}}{2}i$.
11. $\dfrac{\sqrt{2}}{2}$ cis 270° = $-\dfrac{\sqrt{2}}{2}i$.

Exercises XII. E, page 185

1. 125 cis 48° = $83.64 + 92.89i$. **3.** 32 cis 300° = $16 - 16\sqrt{3} \cdot i$.

5. $243 \text{ cis } 140° = -186.1 + 156.2i.$
7. $\text{cis } 0° = 1.$
9. $256 \text{ cis } 0° = 256.$
11. $\frac{1}{16} \text{ cis } 160° = -0.05873 + 0.02138.$
13. $5 \text{ cis } 20° = 4.698 + 1.710i$, $5 \text{ cis } 200° = -4.698 - 1.710i.$
15. $6 \text{ cis } 170° = -5.909 + 1.042i$, $6 \text{ cis } 350° = 5.909 - 1.042i.$
17. $6 \text{ cis } 72° = 1.854 + 5.706i$, $6 \text{ cis } 192° = -5.869 - 1.247i$,
$6 \text{ cis } 312° = 4.015 - 4.459i.$
19. $\sqrt[4]{2} \text{ cis } 22\frac{1}{2}° = 1.0987 + 0.4551i$,
$\sqrt[4]{2} \text{ cis } 202\frac{1}{2}° = -1.0987 - 0.4551i.$
21. $\sqrt[6]{2} \text{ cis } 75° = 0.2905 + 1.0842i$,

$$\sqrt[6]{2} \text{ cis } 195° = -1.0842 - 0.2905i, \quad \sqrt[6]{2} \text{ cis } 315° = \frac{\sqrt[3]{4}}{2}(1 - i).$$

23. $\text{cis } 0° = 1$, $\text{cis } 120° = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$, $\text{cis } 240° = -\frac{1}{2} - \frac{\sqrt{3}}{2} i.$
25. $\sqrt{2} \text{ cis } 9° = 1.3968 + 0.2212i$, $\sqrt{2} \text{ cis } 81° = 0.2212 + 1.3968i$,
$\sqrt{2} \text{ cis } 153° = -1.2601 + 0.6420i$, $\sqrt{2} \text{ cis } 225° = -1 - i$,
$\sqrt{2} \text{ cis } 297° = 0.6420 - 1.2601i.$
27. $\text{cis } 67\frac{1}{2}° = 0.3827 + 0.9239i$, $\text{cis } 157\frac{1}{2}° = -0.9239 + 0.3827i$,
$\text{cis } 247\frac{1}{2}° = -0.3827 - 0.9239i$, $\text{cis } 337\frac{1}{2}° = 0.9239 - 0.3827i.$
29. $\sqrt[3]{5} \text{ cis } 15° = 1.6517 + 0.4426i$, $\sqrt[3]{5} \text{ cis } 75° = 0.4426 + 1.6517i$,

$$\sqrt[3]{5} \text{ cis } 135° = \frac{\sqrt[6]{200}}{2}(-1 + i),$$

$$\sqrt[3]{5} \text{ cis } 195° = -1.6517 - 0.4426i,$$

$$\sqrt[3]{5} \text{ cis } 255° = -0.4426 - 1.6517i, \quad \sqrt[3]{5} \text{ cis } 315° = \frac{\sqrt[6]{200}}{2}(1 - i).$$

31. $6, -3 \pm 3\sqrt{3} \cdot i.$
33. $2(1 \pm i), 2(-1 \pm i).$
35. $\frac{1}{2}(\sqrt{3} \pm i), \frac{1}{2}(-\sqrt{3} \pm i).$
37. $\text{cis } 72° = 0.3090 + 0.9511i$, $\text{cis } 144° = -0.8090 + 0.5878i$,
$\text{cis } 216° = -0.8090 - 0.5878i$, $\text{cis } 288° = 0.3090 - 0.9511i.$

Exercises XIII. A, page 191

In exercises 1–21 the first expression given is the quotient, the second is the remainder.

1. $x^2 + 6x + 10, 15.$
3. $x^2 + 4x, 7.$
5. $x^2 - 2x + 4, 0.$
7. $3x^2 + 12x + 16, 4.$
9. $x^3 + 3x^2 + 4, -1.$
11. $4x^3 + 2x^2 + 4x + 2, 0.$
13. $3x^3 - 9x^2 + 6x - 3, 0.$
15. $3x^2 + 0.7x + 0.51, 0.130.$
17. $x^2 + 4ax - 2a^2, a^3.$
19. $3x^3 - 3\sqrt{2} \cdot x^2 + 2x - 2\sqrt{2}, 0.$
21. $2x^3 - 4ix^2 - 5x + 10i, 0.$
23. $13, -62.$
25. $0, -60.$

Exercises XIII. B, page 195

1. 2, 4, −1.
3. 3, 5, (−1,0), (i.e., between −1 and 0).
5. 3, −1, (−4,−3). **7.** (0,1), (2,3), (−3,−2).
9. −2, (1,2), (4,5). **11.** (−4,−3).
13. −1, (1,2). **15.** No real zeros.
17. 2, −5. **19.** 3, −2. **21.** 4, −4. **23.** 6, −2.

Exercises XIII. C, page 200

3. $-2 \pm i\sqrt{5}, \dfrac{-1 \pm \sqrt{41}}{4}$. **5.** $\pm i, \pm i, 2 \pm \sqrt{10}$.

Exercises XIII. E, page 204

NOTE. (1,0,2), for example, means 1 positive, 0 negative, 2 imaginary.

1. (1,0,2). **3.** (1,2,0), (1,0,2).
5. (2,1,0), (0,1,2). **7.** (2,1,0), (0,1,2).
9. (1,2,0), (1,0,2). **11.** (3,0,0), (1,0,2).
13. (2,1,0), (0,1,2). **15.** (1,2,0), (1,0,2).
17. (0,3,0), (0,1,2). **19.** (0,2,2), (0,0,4).
21. (2,2,0), (2,0,2), (0,2,2), (0,0,4).
23. (2,2,0), (2,0,2), (0,2,2), (0,0,4).
25. (3,1,0), (1,1,2). **27.** (2,2,0), (2,0,2), (0,2,2), (0,0,4).
29. (0,0,4). **31.** (0,1,4). **33.** (2,0,4), (0,0,6).

Exercises XIII. F, page 208

1. 3. **3.** 4. **5.** $\frac{1}{2}$. **7.** $\frac{1}{2}, \frac{2}{3}, -\frac{3}{2}$.
9. 2. **11.** None. **13.** $0, \frac{4}{3}$. **15.** $-\frac{3}{2}, -\frac{3}{2}$.
17. $\frac{2}{5}$. **19.** $4, -5, -\frac{2}{3}, -\frac{3}{2}$. **21.** $4\frac{1}{2}$ in.

Exercises XIII. G, page 211

1. 1.379. **3.** 1.879, −0.347, −1.532.
5. −2.426. **7.** 1.761.
9. 0.340, 2.262, −2.602. **11.** 2.213.
13. 2.729, −2.587. **15.** 1.710.
17. 2.414, −0.414.

Exercises XIII. H, page 213

1. $x^3 - x + 4 = 0$. **3.** $x^3 - 8x + 7 = 0$. **5.** $4x^3 - 7 = 0$.
7. $2x^3 - 3.8x^2 - 7.76x - 8.384 = 0$, or
$x^3 - 1.9x^2 - 3.88x - 4.192 = 0$.
9. $x^4 - 100x^2 - 425x - 150 = 0$.

Exercises XIII. I, page 216

1. 1.328. **3.** −3.379. **5.** 2.491, −0.657, −1.834. **7.** −1.466.
9. 4, 2.180. **11.** 5, $\frac{1}{2}$, $\frac{3}{2}$, $-\frac{2}{3}$.
13. ±0.866, 0.530, −0.419.
15. 1.207, −0.707, −1.586, −4.414.
17. −1. **19.** 3.684. **21.** 1.516.
23. (3,2), (3.584,−1.848), (−2.805,3.131), (−3.779,−3.283).
25. 0.72 in. **27.** 5.434 in., 9.434 in.
29. Diameter 7.710 in., height 10.710 in.
31. 0.420. **33.** 0.472. **35.** 0.805. **37.** 0.544. **39.** 0.494.

Exercises XIII. J, page 225

1. $\sqrt[3]{3} + \sqrt[3]{9}$, $\omega\sqrt[3]{3} + \omega^2\sqrt[3]{9}$, $\omega^2\sqrt[3]{3} + \omega\sqrt[3]{9}$.
3. $4 \cos 40° + 2 = 5.064$, $-4 \cos 20° + 2 = -1.759$, $4 \cos 80° + 2 = 2.695$.
5. $1 + \sqrt[3]{5} + \sqrt[3]{25}$, $1 + \omega\sqrt[3]{5} + \omega^2\sqrt[3]{25}$, $1 + \omega^2\sqrt[3]{5} + \omega\sqrt[3]{25}$.
7. $5, 2 \pm \sqrt{3}$. **9.** $-2 \pm \sqrt{7}$, $-2 \pm i\sqrt{3}$.
11. $2 \pm \sqrt{3}$, $\pm 2i$. **13.** $-1 \pm \sqrt{6}$, $-2 \pm \sqrt{7}$.
15. $2(1 + 5\sqrt{2} \pm \sqrt{47 - 10\sqrt{2}})$ ft.

Exercises XIII. K, page 226

1. −6, −5. **3.** $\frac{3}{2}$, 2. **5.** −4, −9.
7. $\frac{4}{3}$, $\frac{2}{3}$. **9.** $7 \pm \sqrt{5}$, $-\frac{3}{2}$; 46. **11.** $-\frac{1}{3}$, $-\frac{1}{3}$, 2; −20.
13. $\frac{3}{2}, \frac{3}{2}, -\frac{5}{3}, -\frac{5}{3}$, $c = -\frac{179}{3}$, $d = -10$;
$\dfrac{-1 \pm i\sqrt{359}}{12}, \dfrac{-1 \pm i\sqrt{359}}{12}$, $c = \dfrac{181}{3}$, $d = 10$.
15. $\frac{4}{3}, \frac{4}{3}, \frac{4}{3}, -\frac{1}{2}$, $d = \frac{16}{9}$, $e = -\frac{64}{9}$; $\frac{5}{12}, \frac{5}{12}, \frac{5}{12}, \frac{9}{4}$, $d = -\frac{2125}{288}$, $e = \frac{125}{128}$.

Exercises XIV. A, page 232

1. 2. **3.** 4. **5.** −1. **7.** 4. **9.** −4. **11.** 1.
13. −3. **15.** 0. **17.** 0. **19.** −4. **21.** 1. **23.** −1.
25. 728. **27.** 0.0728. **29.** 72800. **31.** 0.000728.
33. 4.23. **35.** 0.0423. **37.** 0.000,042,3. **39.** 4.23.

Exercises XIV. B, page 235

3. 0.4771. **5.** 2.6021. **7.** 9.8007 − 10.
9. 0.2867. **11.** 1.5780. **13.** 4.9024.
15. 8.8688 − 10. **17.** 9.9999 − 10. **19.** 6.9484.
21. 8.0004 − 10.

Exercises XIV. C, page 236

1. 3.520. **3.** 0.2060. **5.** 6.000. **7.** 167.3.
9. 6415. **11.** 31.62. **13.** 10.47. **15.** 1.024.
17. 0.03442. **19.** 3428. **21.** 1.085. **23.** 0.005495.

Exercises XIV. D, page 244

Answers were computed with four-place logarithms. Where correct result (to proper number of significant digits) differs, it is given in ( ).

1. 680.3 (680.4). **3.** 12.86 (12.87). **5.** 0.02930.
7. 0.2229. **9.** 5.499 (5.500). **11.** 8.024 (8.026).
13. 343.8 (343.9). **15.** 0.9135 (0.9136). **17.** 0.5776 (0.5775).
19. 17.04 (17.03). **21.** 0.000,223,5 (0.000,223,4).
23. 0.2249. **25.** 10.69. **27.** 0.7972.
29. 6.480 (6.482). **31.** 2.819. **33.** 1.213.
35. 0.2198 (0.2197). **37.** 0.1902 (1.903). **39.** 0.001430.
41. $-0.1004i$ ($-0.1005i$). **43.** 0.9800 (0.9802).
45. 0.9986 (0.9987). **47.** 0.7632.
49. 0.2531. **51.** 0.5645 (0.5646). **53.** -0.2295.
55. 6.920 (6.919). **57.** 1.016. **59.** 193.8 sq. in.
61. 1611 (1612) cu. in. **63.** 77700 cu. in.
65. 2.048 sec. **67.** (a) 838.7 (838.6), (b) 98.28 (98.33).

Exercises XIV. E, page 246

1. 4.322. **3.** 2.737. **5.** 2.666.
7. -1.431. **9.** 0.7823. **11.** 0.3396.
13. 1.517. **15.** 15. **17.** 41.
19. $-b + \dfrac{\log c}{\log a}$. **21.** 3.134, -0.880. **23.** 3.650, -1.778.
25. 1.664, 1.966. **27.** $\dfrac{\log b \cdot \log c}{\log a\,(\log a + \log b)}, \dfrac{\log c}{\log a + \log b}$.

Exercises XIV. F, page 247

1. 5. **3.** $\frac{1}{16}$. **5.** -2. **7.** $\sqrt{3}$. **9.** $8\sqrt{2}$.
11. $\frac{3}{2}$. **13.** $\frac{1}{3}$. **15.** $\frac{2}{3}$. **17.** $\frac{1}{5}$. **19.** $\frac{1}{4}$.
21. $\dfrac{\sqrt{2}}{4}$. **23.** Any no. $\neq 0$. **25.** $\dfrac{1}{b^3}$.
27. b^b. **29.** $\dfrac{1}{a^2}$.

Exercises XIV. G, page 250

3. 2.322. **5.** 1.609. **7.** 0.6931.
9. 3.096. **11.** 1.074. **13.** 4.568.

15. 2.851. **17.** -1.1754, or $0.8246 - 2$.
19. $\frac{1}{2}$.

Exercises XIV. H, page 252

1. 1.5723, 6.5072. **3.** 2.5060. **5.** 1, 0.8259.
7. 0.4263. **9.** 2, 4, -0.7667. **11.** 0.0210.

Exercises XV. A, page 256

Answers given first are from tables in book. Those in ( ) are correct to nearest cent; they are omitted when book tables are adequate.

1. (a) \$121.67, (b) \$121.90, (c) \$122.02.
3. \$112.68. **5.** \$1160.32 (\$1160.33).
7. 11. **9.** 5%. **11.** 2.5%. **13.** 4%.
15. (a) 11.9 yr., (b) 11.7 yr., (c) 11.6 yr., (d) 11.4 yr. (4-place logs), 11.6 yr. (7-place logs).
17. $8\frac{1}{2}$ yr.
19. (a) \$102.00 (\$101.98), (b) \$101.00 (\$100.99), (c) \$100.30 (\$100.33). **21.** 256000.

Exercises XV. B, page 262

1. \$5416.30 (\$5416.32). **3.** \$2292.78.
5. \$6356.10 (\$6356.12). **7.** \$906.54 (\$906.53).
9. \$97122 (\$97122.49). **11.** \$7358.95 (\$7358.94).
13. \$452.28 (\$452.29). **15.** \$6243.20 (\$6243.18).
17. \$8390.75 (\$8390.70). **19.** \$5438.70 (\$5438.67).
21. 4.17%.

Exercises XVI. A, page 268

1. 720. **3.** 72. **5.** 12.
7. Every 2 mi. **9.** 999999. **11.** (a) 180, (b) 648.
13. 1840. **15.** (a) 241920, (b) 967680.
17. 210. **19.** (a) 443520, (b) 169344, (c) 112896.

Exercises XVI. B, page 272

1. 35. **3.** 28. **5.** 190. **7.** 120.
9. 16002. **11.** 20,358,520. **13.** 127. **15.** 945.
17. 302400. **19.** 106302.

Exercises XVI. C, page 274

1. 12. **3.** 720. **5.** 40320. **7.** 15. **9.** 56.
11. 10080. **13.** 220. **15.** 3360. **17.** 144. **19.** 11880.
21. 325. **23.** 42. **25.** 55986. **27.** 252. **29.** 10240.
31. (a) 53, (b) 758.

Exercises XVII. A, page 279

1. (a) $\frac{3}{5}$, (b) $\frac{2}{5}$.
3. $\frac{1}{3}$.
5. $12\frac{1}{2}$¢.
7. \$0.62$\frac{1}{2}$.
9. (a) 0.0078 (b) 0.9922.
13. (a) 0.8297, (b) 0.1703.
15. 0.0261.
17. (a) $\frac{22}{51}$, (b) $\frac{5}{51}$, (c) $\frac{8}{17}$.
19. $\frac{1}{55}$.
21. $\frac{52}{161}$.
23. $\frac{1}{21}$.

Exercises XVII. B, page 285

1. $\frac{4}{5}$.
3. $\frac{2}{3}$.
5. 0.2634.
7. (a) $\frac{1}{8}$, (b) $\frac{8}{2197}$, (c) $\frac{27}{2197}$.
9. (a) 0.000,153,8, (b) 0.01035.
11. (a) $\frac{1}{16}$, (b) $\frac{3}{169}$, (c) $\frac{1}{169}$.
13. $\frac{5}{8}$.

Exercises XVII. C, page 288

1. (a) $\frac{5}{16}$, (b) $\frac{1}{2}$.
3. (a) $\frac{625}{11664}$, (b) $\frac{1453}{23328}$.
5. (a) 0.107, (b) 0.376.
7. (a) 0.00208, (b) 0.00069, (c) 0.99998.
9. (a) 0.531441, (b) 0.098415, (c) 0.114259.

Exercises XVII. D, page 289

1. (a) $\frac{4}{9}$, (b) $\frac{28}{55}$.
3. $\frac{5}{36}$.
5. $\frac{1}{25}$.
7. $\frac{1}{3}$.
9. (a) $\frac{1}{4096}$, (b) $\frac{231}{1024}$, (c) $\frac{299}{4096}$.
11. A, \$2$\frac{6}{7}$; B, \$2$\frac{1}{7}$.
13. A, $\frac{1}{2}$; B, $\frac{1}{2}$.
15. $\frac{1}{4}$.
17. $\frac{1}{7}$.
19. 3.84¢.
21. (a) $\frac{1}{10}$, (b) 0, (c) $\frac{3}{5}$, (d) 0, (e) $\frac{3}{10}$, (f) 0.
23. (a) $\frac{15}{64}$, (b) 0.
25. $\left(\frac{n-2}{n-1}\right)^{n-1}$.

Exercises XVIII. B, page 299

1. -11.
3. 180.
5. -23343.
7. -6146.
9. $-\frac{1}{120}$.
11. $-4, \frac{3}{2}$.
13. 2, 7, -5.
15. -3, 9, 4.
17. 2, 2, 2.
19. $\frac{197}{719}, \frac{40}{719}, \frac{145}{719}$.
21. 12, 4, 10.

Exercises XVIII. C, page 307

1. 36960.
3. 0.
5. -17.
7. 0.
9. -27.
11. 286.
13. 1872.
15. 902.

Exercises XVIII. D, page 316

1. $x = -1, y = 2, z = -3, t = 4$.
3. $x = 5, y = -7, z = -2, t = 3$.
5. $A = 3, B = 1, C = 2, D = -1$.
7. $x = 7, y = 4, z = -3, t = 5, w = -1$.

9. $x = \frac{3c - 28}{3}$, $y = \frac{-6c + 29}{3}$, $z = c$, c arbitrary.

11. $x = 2c + 1$, $y = -c + 3$, $z = c$, c arbitrary.

13. Inconsistent.

15. $x = \frac{-3c + 2}{5}$, $y = \frac{-2c + 3}{5}$, $z = c$, c arbitrary.

17. $x = 5$, $y = 8$.

19. Inconsistent.

21. $x = 2$, $y = -7$.

23. $x = -1$, $y = 2$.

25. $x = 2$, $y = 3$, $z = -1$.

27. No nontrivial solution.

29. $x : y : z = -9 : 4 : 6$.

31. $x : y : z = 1 : -3 : 8$.

Exercises XIX. A, page 326

1. $\frac{3}{x - 1} + \frac{2}{x - 2}$.

3. $\frac{1}{x - 6} - \frac{1}{x - 5}$.

5. $\frac{3}{2(x - 2)} - \frac{1}{2(x + 2)}$.

7. $\frac{5}{x - 4} - \frac{3}{2x - 3}$.

9. $\frac{6}{2x + 3} - \frac{7}{5x - 1}$.

11. $\frac{4}{x - 4} + \frac{5}{x - 2}$.

13. $\frac{18}{x - 6} - \frac{7}{x - 3}$.

15. $\frac{1}{x - 6} - \frac{6}{12x + 1}$.

17. $\frac{3}{x + 1} - \frac{2}{x - 2} + \frac{5}{x - 4}$.

19. $\frac{6}{x + 1} + \frac{3}{x - 6} - \frac{2}{x}$.

21. $\frac{3}{(x - 3)^2} + \frac{2}{x - 3}$.

23. $\frac{2}{x + 1} + \frac{2}{x - 3} - \frac{5}{(x - 3)^2}$.

25. $\frac{2}{x - 2} - \frac{4}{2x + 1} + \frac{5}{(2x + 1)^2}$.

27. $\frac{1}{x - 1} - \frac{x - 4}{x^2 + 4}$.

29. $\frac{16x + 9}{11(x^2 - 3x + 5)} - \frac{10}{11(2x - 3)}$.

31. $\frac{1}{x + 2} + \frac{1}{x - 2} + \frac{2x}{x^2 + 4}$.

33. $\frac{x + 1}{3(x^2 - x + 1)} - \frac{1}{3(x + 1)}$.

35. $\frac{x}{x^2 + 3} - \frac{3x}{(x^2 + 3)^2}$.

37. $\frac{1}{x - 2} - \frac{x + 2}{x^2 + 2} - \frac{6x + 12}{(x^2 + 2)^2}$.

39. $\frac{4}{x + 2} + \frac{3}{(x + 2)^2} - \frac{4x - 5}{x^2 + 2} - \frac{12x - 6}{(x^2 + 2)^2}$.

41. $\frac{1}{x - 1} - \frac{2}{(x - 1)^2} + \frac{1}{(x + 1)^2} + \frac{5x}{x^2 + 1} - \frac{2}{(x^2 + 1)^2}$.

43. $x + 4 + \frac{3}{x-2} - \frac{1}{x+5}$. **45.** $3x + 2 + \frac{3}{2x+1} - \frac{4}{x-4}$.

47. $2 - \frac{7}{16(3x+2)} - \frac{147}{16(x+2)} + \frac{47}{4(x+2)^2}$.

49. $x + \frac{2}{2x+3} + \frac{x-3}{2x^2+3}$.

51. $x^2 + 2x + 1 + \frac{5}{3x} - \frac{17x-1}{3(x^2-2x+3)}$.

Exercises XX. A, page 329

1. $\frac{1}{3n}$. **3.** $\frac{9}{8}\left(\frac{2}{3}\right)^{n-1}$. **5.** $\frac{1}{n^2}$. **7.** $\frac{n}{(n+1)^2}$.

9. $\frac{1}{(4n-3)(4n-1)}$. **11.** $\frac{\sqrt[3]{n}}{n}$.

13. $\frac{1}{n!}$. **15.** $\frac{2^n}{n!}$.

17. $\frac{1}{(n+1)^2 - n}$. **19.** $\frac{1}{1 - 0.01n}$.

21. $\frac{1}{(1 + 0.1n)^2}$. **23.** $\frac{1}{\log_{10}(n+1)}$. **25.** $\log_{10}\frac{1}{2^n}$.

Exercises XX. B, page 339

C means convergent, D means divergent.

1. D. **3.** C. **5.** C. **7.** D. **9.** C. **11.** D. **13.** C.
15. C. **17.** C. **19.** D. **21.** C. **23.** D. **25.** D.

Exercises XX. C, page 346

1. C. **3.** D. **5.** C. **7.** D. **9.** D. **11.** C. **13.** D.
15. C. **17.** D.

Exercises XX. D, page 348

1. $-1 \leqq x < 1$. **3.** $-\infty < x < \infty$. **5.** $-1 < x < 1$.
7. $-1 \leqq x \leqq 1$ **9.** $-1 \leqq x \leqq 1$. **11.** $-1 < x < 1$.
13. $-1 < x < 3$. **15.** $-1 \leqq x \leqq 1$. **17.** $x = 0$.
19. $a - b < x < a + b$. **21.** $-1 < x < 1$.

Exercises XXI. A, page 358

1. 1st differences: 3, 5, 7, 9; 2nd differences const. = 2.
3. 1st differences: 3, 11, 25, 45, 71; 2nd differences: 8, 14, 20, 26; 3rd differences const. = 6.

5. 1st differences: $-2, 6, 44, 136, 306, 578, \cdots$; 2nd differences: 8, 38, 92, 170, 272; 3rd differences: 30, 54, 78, 102; 4th differences const. $= 24$.

7. $-41, -m^2 + 9m - 5$.

9. $168, -\frac{1}{24}(3m^4 - 14m^3 - 231m^2 + 458m + 1080)$.

11. 0.1864.

13. 2.9300.

15(1). $395, \frac{1}{16}m(2m^2 + 3m + 7)$.

15(3). $2365, \frac{1}{12}m(3m^3 - 2m^2 + 3m + 8)$.

15(5). $13118, \frac{1}{60}m(12m^4 - 45m^3 + 50m^2 - 75m - 542)$.

17. $\frac{1}{4}m^2(m + 1)^2$.

19. $\frac{1}{3}m(4m^2 - 1)$.

21. $\frac{1}{12}m(m + 1)(3m^2 - 5m + 14)$.

23. $\frac{1}{2}m(m^3 + 2m - 7)$.